CHARACTERIZATIONS OF C^*-ALGEBRAS

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This note presents two related characterizations of those Banach algebras which are isometrically isomorphic to C^*-algebras, i.e., to operator-norm closed, self-adjoint algebras of operators on a Hilbert space. The first characterization has evolved from a theorem of I. Vidav [7] and its extension by E. Berkson [1] and B. W. Glickfeld [2] (cf. [5]). The proof given below is considerably simpler than the proofs given in [1] and [2] for closely related, but weaker, results. It is based on Lemma 1 which refines a result of B. Russo and H. A. Dye [6].

All algebras considered here have complex scalars and an identity element I of norm one.

In [6] it is shown that the closed unit ball $\mathfrak{A}_1$ of a C^*-algebra $\mathfrak{A}$ is the norm closed convex hull clco $U(\mathfrak{A})$ of the set $U(\mathfrak{A})$ of all unitary elements in $\mathfrak{A}$. The set $\mathfrak{A}_e = \{e^{iR}: R \in \mathfrak{A}, R = R^*\}$, which can be defined in any Banach algebra with an involution, is a subset of $U(\mathfrak{A})$ in any C^*-algebra. In a von Neumann algebra $\mathfrak{A}_e = U(\mathfrak{A})$, but in certain C^*-algebras $\mathfrak{A}_e$ is a proper subset of $U(\mathfrak{A})$. For instance if $\mathfrak{A}$ is the usual Banach algebra of continuous functions on the unit circle in the complex plane, then multiplication by the complex variable belongs to $U(\mathfrak{A})$ but not to $\mathfrak{A}_e$. Thus the following lemma strengthens Theorem 1 of [6].

LEMMA 1. *If $\mathfrak{A}$ is a C^*-algebra,* clco $\mathfrak{A}_e = \mathfrak{A}_1$.

PROOF. A unitary element in a C^*-algebra belongs to $\mathfrak{A}_e$ if its spectrum does not include the whole unit circle. Thus the proof of Theorem 2 in [3] gives as a special case: If $\mathfrak{A}$ is a C^*-algebra and $(^s)$ represents closure in the strong operator topology, then $(\mathfrak{A}_e)^s \supseteq U(\mathfrak{A}^s)$. Using this statement, the proof of Theorem 1 in [6] now proves this lemma.

DEFINITION. (Cf. [4], [5], [7].) For any Banach algebra $\mathfrak{A}$, let

$$\mathfrak{A}_s = \{R \in \mathfrak{A}: \|e^{itR}\| = 1 \quad \text{for all real } t\}.$$

Alternatively $\mathfrak{A}_s$ could be described as the set of elements in $\mathfrak{A}$ with real numerical range ([4], [5]) in some (hence any) isometric repre-

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sentation of $\mathfrak{A}$ on a Banach space, or as the set of elements R in $\mathfrak{A}$ for which

$$\lim_{t\to 0} \frac{\|I + itR\| - 1}{t} = 0, \qquad t \text{ real.}$$

Clearly if $\mathfrak{A}$ is isometrically isomorphic to a C^*-algebra, then $\mathfrak{A}_s$ corresponds to the set of self-adjoint elements.

THEOREM. *A Banach algebra $\mathfrak{A}$ is isometrically isomorphic to a C^*-algebra if and only if one (hence both) of the following conditions hold:*

(1) *For each $S\in\mathfrak{A}$ there is at least one pair R, $J\in\mathfrak{A}_s$ with $S=R+iJ$.*

In this case R and J are uniquely determined by S and $\pi(R-iJ)=\pi(R+iJ)^*$ where π is any isometric representation of $\mathfrak{A}$ as a C^*-algebra.

(2) *There is a real linear subspace $\mathfrak{A}_r$ of $\mathfrak{A}$ such that $R^2\in\mathfrak{A}_r$ if $R\in\mathfrak{A}_r$ and $\mathfrak{A}_1=\mathrm{clco}\{e^{iR}: R\in\mathfrak{A}_r\}$.*

In this case the norm closure of the real linear span of $\mathfrak{A}_r\cup\{I\}$ is $\mathfrak{A}_s$ and (1) *holds.*

The first condition can be informally stated: A Banach algebra $\mathfrak{A}$ is a C^*-algebra iff $\mathfrak{A}=\mathfrak{A}_s+i\mathfrak{A}_s$. It differs from Corollary 4.4 of [1] in the omission of a hypothesis.

PROOF. Condition (1) is necessary since e^{iR} is unitary if R is self-adjoint. Lemma 1 shows the necessity of (2) when the set of self-adjoint elements is chosen as $\mathfrak{A}_r$.

Assume next that condition (1) is satisfied. Then the decomposition $S=R+iJ$ is unique [7, Hilfssatz 2c]. For $S\in\mathfrak{A}_s$ let $S^2=R+iJ$ with R, $J\in\mathfrak{A}_s$. Then $(R+iJ)S=S(R+iJ)$ so $(RS-SR)=i(SJ-JS)$. However both $i(RS-SR)$ and $i(SJ-JS)$ belong to $\mathfrak{A}_s$ [7, Hilfssatz 2b], so they are both zero, and R commutes with S, S^2 and hence with J. Therefore $\mathfrak{A}$ satisfies the hypotheses of Vidav's theorem [7] and the proof of the sufficiency of (1) could be concluded by citing results in [1]. However the proof can also be completed without reference to the arguments of §§2 and 3 of [1].

Vidav's theorem exhibits a norm $\|\cdot\|_0$ on $\mathfrak{A}$ equivalent to the given norm $\|\cdot\|$ on $\mathfrak{A}$ and equal to it on $\mathfrak{A}_s$, and shows that the map $R+iJ\to R-iJ$ is an involution on $\mathfrak{A}$, relative to which $\mathfrak{A}$, normed by $\|\cdot\|_0$, is isometrically $*$-isomorphic to a C^*-algebra. Thus Lemma 1 shows that the closed unit ball of $\mathfrak{A}$ relative to $\|\cdot\|_0$ is clco $\mathfrak{A}_e$ which is certainly a subset of the closed unit ball of $\mathfrak{A}$ relative to $\|\cdot\|$. Therefore $\|S\|\leqq\|S\|_0$ for all $S\in\mathfrak{A}$. However if $\|S\|<\|S\|_0$ then

$$\|S^*S\| \leqq \|S^*\|\,\|S\| < \|S^*\|_0\|S\|_0 = \|S^*S\|_0 = \|S^*S\|.$$

This contradiction proves that $\mathfrak{A}$ is already isometrically isomorphic to a C^*-algebra under its original norm. Furthermore if π is any isometric representation, then the inverse image of the set of self-adjoint elements is a subset of $\mathfrak{A}_s$. Thus the uniqueness of the decomposition $S=R+iJ$ proves that $\pi(R-iJ)=\pi(R+iJ)^*$.

Finally if (2) holds, we may assume $\mathfrak{A}_r$ contains I and is closed in the norm topology since the norm closure of the real linear span of $\mathfrak{A}_r \cup \{I\}$ shares the defining properties of $\mathfrak{A}_r$. Furthermore if $R \in \mathfrak{A}_r$, then by induction $\mathfrak{A}_r$ contains any positive power of R since

$$R^{m+2n} = (1/2)[(R^m + R^{2n})^2 - (R^m)^2 - R^{2n+1}].$$

Thus $\mathfrak{A}_r$ contains $\cos(R)$ and $\sin(R)$ and the norm limit of a convex combination of such elements. Since $\mathfrak{A}_r \subseteq \mathfrak{A}_s$, condition (2) implies condition (1). The uniqueness of the decomposition $S=R+iJ$ shows that $\mathfrak{A}_r$ (as expanded above) is $\mathfrak{A}_s$.

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