BOUNDEDNESS OF SOLUTIONS TO LINEAR DIFFERENTIAL EQUATIONS

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In the case of a linear constant coefficient differential equation, $\dot{x}=Ax$, where x is a (complex) n-vector and A is a (complex) $n\times n$ matrix, it is well known when all solutions are bounded; namely, if all eigenvalues of A are purely imaginary and all elementary divisions of A are simple. This condition is equivalent to the Jordan normal form, J, of A being (Hermitian) skew symmetric. That is if $J=PAP^{-1}$, then

$$J + J^* = PAP^{-1} + P^{*-1}A^*P^* = 0, \tag{1}$$

where M^* denotes the adjoint or complex conjugated transpose of M. Multiplying (1) on the left by P^* and on the right by P yields the equivalent condition that there exist a positive definite $Q=P^*P$ such that

$$QA + A^*Q = 0. \tag{2}$$

In the time dependent case, it is shown here that a necessary and sufficient condition that all solutions of $\dot{x}=A(t)x$ be bounded is that there exist a $Q(t)$ that is uniformly bounded and uniformly positive definite and that satisfies

$$Q(t)A(t) + A^*(t)Q(t) + Q(t) = 0.$$

We will use the following notation. If ξ and η are complex n-vectors, then

$$\langle \xi, \eta\rangle = \sum_{i=1}^{n} \xi_i\bar{\eta}_i$$

will denote the inner product of ξ and η, and

$$||\xi|| = \langle \xi, \xi\rangle^{1/2}$$

will denote the norm of ξ. Also M^* will denote the adjoint matrix of a matrix M.

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THEOREM 1. *Let $A(t)$ be an $n\times n$ matrix function defined and continuous on an open interval I. If there exists a continuously differentiable* *matrix function* $Q(t)$ *and real function* $\gamma(t)$ *and* $\delta(t)$ *defined on* I *such that*

$$Q(t)A(t) + A^*Q(t) + \dot{Q}(t) = 0, \tag{3}$$

$$\gamma(t)\|\xi\|^2 \leqq \langle Q(t)\xi, \xi\rangle \leqq \delta(t)\|\xi\|^2, \tag{4}$$

$$0 < \gamma(t) \leqq \delta(t) < \infty, \tag{5}$$

for all t *in* I; *then all solutions of*

$$\dot{x}(t) = A(t)x(t) \tag{6}$$

satisfy

$$\gamma(t)\|x(t)\|^2 \leqq \delta(t_0)\|x(t_0)\|^2 \tag{7}$$

for all t, t_0 *in* I.

Conversely, if there exists a real function $\alpha(t, t_0)$ *defined in* $I \times I$ *such that*

$$\|x(t)\|^2 \leqq \alpha(t, t_0)\|x(t_0)\|^2, \tag{8}$$

$$0 < \alpha(t, t_0) < \infty \tag{9}$$

for all t, t_0 *in* I *and any solution* $x(t)$ *of* (6), *then there exist Hermitian symmetric matrices* $Q_\tau(t)$ *defined for* t *in* I *and satisfying* (3) *for each* τ *in* I. *Also*

$$\alpha^{-1}(t, \tau)\|\xi\|^2 \leqq \langle Q_\tau(t)\xi, \xi\rangle \leqq \alpha(\tau, t)\|\xi\|^2. \tag{10}$$

PROOF. Suppose Q, γ, and δ exist satisfying (3), (4), and (5), and let $x(t)$ be any solution of (6). Then

$$\begin{aligned}(d/dt)&\langle Q(t)x(t), x(t)\rangle \\ &= \langle \dot{Q}(t)x(t), x(t)\rangle + \langle Q(t)\dot{x}(t), x(t)\rangle + \langle Q(t)x(t), \dot{x}(t)\rangle \\ &= \langle [\dot{Q}(t) + Q(t)A(t) + A^*(t)Q(t)]x(t), x(t)\rangle = 0\end{aligned}$$

since (3) holds. Thus $\langle Q(t)x(t), x(t)\rangle$ is a constant, and

$$\begin{aligned}\gamma(t)\|x(t)\|^2 &\leqq \langle Q(t)x(t), x(t)\rangle \\ &= \langle Q(t_0)x(t_0), x(t_0)\rangle \leqq \delta(t_0)\|x(t_0)\|^2.\end{aligned}$$

Now suppose (8) and (9) hold for all solutions of (6). Let $X(t, t_0)$ be the fundamental matrix of (6) such that $X(t_0, t_0) = I$, and define $Q_\tau(t)$ by

$$Q_\tau(t) = X^*(\tau, t)X(\tau, t).$$

It is clear that $Q_\tau(t)$ is defined and differentiable for t in I and all τ in I. Also

$$\begin{aligned} \dot{Q}_\tau(t) &= \dot{X}^*(\tau, t)X(\tau, t) + X^*(\tau, t)\dot{X}(\tau, t) \\ &= -A^*(t)X^*(\tau, t)X(\tau, t) - X^*(\tau, t)X(\tau, t)A(t) \\ &= -[Q_\tau(t)A(t) + A^*(t)Q_\tau(t)], \end{aligned}$$

which is equation (3). We use here the fact that

$$(d/dt)X(\tau, t) = \dot{X}(\tau, t) = -X(\tau, t)A(t).$$

Clearly $Q_\tau(t)$ is Hermitian symmetric. Also

$$\begin{aligned} \langle Q_\tau(t)\xi, \xi\rangle &= \langle X^*(\tau, t)X(\tau, t)\xi, \xi\rangle \\ &= \|X(\tau, t)\xi\|^2 \leqq \alpha(\tau, t)\|\xi\|^2, \end{aligned}$$

by (8) since $X(\tau, t)\xi$ is a solution of (6) (as a function of τ) and $X(t, t)\xi = \xi$. Similarly, we have

$$\begin{aligned} \|\xi\|^2 &= \|X(t, \tau)X(\tau, t)\xi\|^2 \\ &\leqq \alpha(t, \tau)\|X(\tau, t)\xi\|^2 = \alpha(t, \tau)\langle Q_\tau(t)\xi, \xi\rangle. \qquad \text{Q.E.D.} \end{aligned}$$

REMARK. We could have obtained the properties desired for $Q_\tau(t)$ by setting

$$Q_\tau(t) = X^*(\tau, t)RX(\tau, t) \tag{11}$$

where R is any positive definite constant matrix. Observe that we have

$$X^*(t, t_0)Q_\tau(t)X(t, t_0) = Q_\tau(t_0)$$

which can be shown directly from (11) or from (3) and differentiation.

REMARK. If we make a change of variables in (6) by letting

$$y(t) = P(t)x(t),$$

then the differential equation for $y(t)$ becomes

$$\dot{y}(t) = [P(t)A(t)P^{-1}(t) + \dot{P}(t)P^{-1}(t)]y(t) = B(t)y(t).$$

If for some $P(t)$ the resulting $B(t)$ is skew symmetric, then $Q(t) = P^*(t)P(t)$ will satisfy (3) as was shown for the constant coefficient case in the introduction. Further, if there is a differentiable $P(t)$ such that $Q(t) = P^*(t)P(t)$, then the corresponding $B(t)$ is skew symmetric.

COROLLARY. *If in Theorem* 1, $I = (-\infty, \infty)$, *then all solutions of* (6) *are bounded* (*in the sense of* (8) *with* α *a constant*) *if and only if there exists a* $Q(t)$ *satisfying* (3), (4), *and* (5), *but with* γ *and* δ *constant.*

THEOREM 2. *Under the assumptions of Theorem* 1 *there is a constant matrix* $\bar{Q}$ *satisfying* (3), (4), *and* (5) (*with constant* α *and* δ) *if and only if the mean value*

$$\tilde{Q} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} X^*(\tau, t) R X(\tau, t) d\tau$$

exists (boundedly), is independent of t, and is nonsingular for some positive definite matrix R.

PROOF. If Q exists and is constant, then $X^*(\tau, t)QX(\tau, t) = Q$, and $\tilde{Q} = Q$. If $\tilde{Q}$ exists and is independent of t, then

$$\begin{aligned} X^*(t, t_0)\tilde{Q}X(t, t_0) &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} X^*(t, t_0) X^*(\tau, t) R X(\tau, t) X(t, t_0) d\tau \\ &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} X^*(\tau, t_0) R X(\tau, t_0) d\tau = \tilde{Q}. \end{aligned}$$

This is equivalent to (3) by the remark made after Theorem 1. $\tilde{Q}$ is clearly symmetric and nonnegative definite, but it is assumed nonsingular, so it must be positive definite. Q.E.D.

REMARK. If A is constant, then $X(t, t_0) = X(t - t_0) = \exp A(t - t_0)$. Under our assumption that $\exp A\tau$ be bounded, it must be uniformly almost periodic. Thus

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} X^*(\tau, t) X(\tau, t) d\tau$$

exists and is independent of t. Thus there is a constant Q as we know there must be. However, if $X(t, t_0)$ is periodic, there need not be a constant Q satisfying (3) as can be seen by considering a scalar equation.

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