# A CHARACTERIZATION OF SPACES WITH VANISHING GENERALIZED HIGHER WHITEHEAD PRODUCTS

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A subject of recent investigation in homotopy theory has been the study of generalized higher order Whitehead products, cf. [1] and [3]. Let us say that a space, $X$, has property $P_n$ if for any $f_1, \cdots, f_n$, $f_i: SA_i \to X$, we have $0 \in [f_1, \cdots, f_n]$, where $[f_1, \cdots, f_n]$ denotes the set of all $n$th order Whitehead products of $f_1, \cdots, f_n$, as defined in [3]. Thus $0 \in [f_1, \cdots, f_n]$ means that $f_1 \vee \cdots \vee f_n: SA_1 \vee \cdots \vee SA_n \to X$ can be extended to some $F: SA_1 \times \cdots \times SA_n \to X$. (We note at this point that it is an unresolved conjecture as to whether $X$ has property $P_n$ implies that 0 is the *only* element of $[f_1, \cdots, f_n]$.) Now if $X$ is an $H$-space, then $X$ possesses property $P_n$ for all $n$, [3]. Thus multiplicative properties of $X$ itself are too strong to distinguish among the various properties $P_n$. On the other hand, it follows from results of [1] and [4] that a space has property $P_2$ if and only if its loop space is homotopy-commutative. In Theorem 1 below, we shall extend this result to characterize those spaces which have property $P_n$ in terms of higher homotopy-commutativity properties of their loop spaces. Since we shall wish to be able on occasion to replace a loop space by an equivalent CW-monoid, we shall restrict our attention to the category of countable CW-complexes.

The higher homotopy-commutativity properties we need are described in the following definition which was introduced in [7].

DEFINITION. An associative $H$-space, $Y$, is called a $C_n$-*space* provided that there exist maps $Q_i: C(i-1) \times Y^i \to Y$, $1 \leq i \leq n$, such that:

(1) $Q_1: C(0) \times Y \to Y$ is the identity;

(2) $Q_i([\mu, \nu] \circ d_p(r, s), y_1, \cdots, y_i) = Q_p(r, y_{\mu(1)}, \cdots, y_{\mu(p)}) \cdot Q_q(s, y_{\nu(1)}, \cdots, y_{\nu(q)})$, for $(p, q)$-shuffles $(\mu, \nu)$, where $p+q=i$, $r \in C(p-1)$, and $s \in C(i-p)$; and

(3) if $e$ denotes the identity of $Y$, and if $y_i = e$, then

$$Q_i(T, y_1, \cdots, y_i) = Q_{i-1}(D_j(T), y_1, \cdots, \hat{y}_j, \cdots, y_i).$$

Here $C(i)$ is the convex linear cell described in [2], namely the convex hull of the orbit of the point $(1, \cdots, n+1)$ under permutation of the coordinates in $R^{n+1}$. The map $d_p: C(p-1) \times C(i-p) \to C(i)$ is given by $d_p(x_1, \cdots, x_p, y_1, \cdots, y_{i-p+1}) = (x_1, \cdots, x_p, y_1+p, \cdots, y_{i-p+1}+p)$,

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the map $D_j$: $C(i)\to C(i-1)$ is as defined in [2], and $[\mu, \nu]$: $C(i)\to C(i)$ is induced by the actions of the shuffle $(\mu, \nu)$ on $R^{n+1}$ by permutation of coordinates.

Note that a $C_2$-space is just a homotopy-commutative monoid. Furthermore, the usual proof that the loop space of an $H$-space is homotopy-commutative extends to yield the fact that such a loop space is a $C_n$-space for every $n$. (It is known that the converse of this fact is false.) We recall from [7] the main theorem on $C_n$-spaces, which will be used in the proof of Theorem 1.

THEOREM 0. *An associative H-space, Y, is a $C_n$-space if and only if the Hopf fibration for Y, $p_1$: $Y * Y\to SY$, extends to a fibration $p_n$: $E_n \to (SY)_n$, where $(SY)_n$ denotes the n-fold reduced product space of the suspension of Y.*

The idea of $C_n$-commutativity is somewhat analogous to Stasheff's theory of $A_n$-associativity, [5]. Thus the reduced product spaces $(SY)_n$ stand in relation to commutativity much as the projective spaces $XP(n)$ relate to associativity. The proof of the "only if" part of Theorem 0 is inspired by Stasheff's work, and is accomplished by a direct construction in the Dold-Lashof vein. The reverse implication uses the connecting map $r$: $\Omega(SY)_n\to Y$ together with the fact that $Y$ is $C_n$ in $\Omega(SY)_n$. Here $Y$ is regarded as a subspace of $\Omega(SY)_n$ via the composition $Y\xrightarrow{j}\Omega SY\subset\Omega(SY)_n$, where $j$ is the usual inclusion. The notion of a subspace being $C_n$ in a containing space is a natural extension of the well-known concept of a subspace being homotopy-commutative in a larger space.

The definition of $C_n$-space permits us to state the main theorem of this note.

THEOREM 1. *A space possesses property $P_n$ if and only if its loop space is a $C_n$-space.*

The proof of this theorem, which will be outlined below, is based on the following theorem.

THEOREM 2. *A monoid, Y, is a $C_n$-space if and only if the inclusion $i$: $SY\to B_Y$ extends to a map $a$: $(SY)_n\to B_Y$.*

The main theorem follows readily from Theorem 2 as follows. Let $X$ be a space and $Y$ a monoid for which there exists a strongly homotopy-multiplicative homotopy equivalence $f$: $Y\to\Omega X$, as in [6]. Then $f$ induces a homotopy equivalence $g$: $B_Y\to X$. Let $c$ be the evaluation map $c$: $S\Omega X\to X$. Then $(Sf, g)$ is a homotopy equivalence of the map $i$ with the map $c$. Now any map $h$: $SA\to X$ factors as $c\cdot Sh$, where $\bar{h}$ is the adjoint of $h$, and hence $h$ factors up to homotopy through $i$. Now if $\Omega X$ is a $C_n$-space, then so is $Y$, and hence $0 \in [i, \cdots, i]$ ($n$ factors), by Theorem 2. Consequently if $f_i\colon SA_i \to X$, $1 \leqq i \leqq n$, then $0 \in [f_1, \cdots, f_n]$, and thus $X$ satisfies property $P_n$. Conversely, it follows from Theorem 2.8 of [3] that $0 \in [i, \cdots, i]$ implies that $i$ extends to all of $(SY)_n \to B_Y$.

The proof of Theorem 2 goes as follows. The "if" part is easily obtained from Theorem 0 by taking $p_n\colon E_n \to (SX)_n$ to be the fibration induced by $a\colon (SY)_n \to B_Y$ from $\pi\colon \mathcal{E}_\infty \to B_Y$, the Dold-Lashof universal fibration for $Y$. The converse is the nontrivial implication and is accomplished by using the maps $Q_i$ to map $E_n$ to $\mathcal{E}_{n+1}$, the total space of the $(n+1)$th stage of the Dold-Lashof construction, in fiber-wise fashion, thus inducing a map in the base spaces $(SY)_n \to YP(n) \subset B_Y$. The details are rather lengthy and will appear elsewhere.

## References

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