INTRINSIC CHARACTERIZATION OF POLYNOMIAL TRANSFORMATIONS BETWEEN VECTOR SPACES OVER A FIELD OF CHARACTERISTIC ZERO[1]

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1. Introduction. Examples. A complex valued function u of a complex argument is a polynomial function $u(z)=\mathfrak{a}z^3+\mathfrak{b}z^2+\mathfrak{c}z+\mathfrak{d}$ of degree at most three if and only if u satisfies the inhomogeneous inclusion-exclusion identity of degree three

$$\begin{aligned} &u(\beta+\gamma+\mathfrak{d}\delta)-u(\beta-\gamma+\mathfrak{d}\delta)-u(\beta+\gamma-\mathfrak{d}\delta)+u(\beta-\gamma-\mathfrak{d}\delta)\\ &\quad=\mathfrak{d}(u(\beta+\gamma+\delta)-u(\beta-\gamma+\delta)-u(\beta+\gamma-\delta)+u(\beta-\gamma-\delta)), \end{aligned}$$

for all complex numbers β, γ, δ, $\mathfrak{d}$. The function $u(z)=z+1$ is a polynomial function of degree at most three. Suppose a real valued function t of two real arguments is Euler homogeneous of degree three. Then t is a cubic form $t(x,y)=\mathfrak{e}x^3+\mathfrak{f}x^2y+\mathfrak{g}xy^2+\mathfrak{h}y^3$ if and only if either t satisfies the heterogeneous inclusion-exclusion identity of degree three

$$\begin{aligned} &(t(\beta+\gamma+\mathfrak{d}\delta)-t(-\beta+\gamma+\mathfrak{d}\delta)-t(\beta-\gamma+\mathfrak{d}\delta)-t(\beta+\gamma-\mathfrak{d}\delta))/24\\ &\quad=\mathfrak{d}(t(\beta+\gamma+\delta)-t(-\beta+\gamma+\delta)-t(\beta-\gamma+\delta)-t(\beta+\gamma-\delta))/24, \end{aligned}$$

for all ordered pairs β, γ, δ of real numbers, all real numbers $\mathfrak{d}$, or t satisfies the homogeneous inclusion-exclusion identity of degree three

$$\begin{aligned} &(t(\mathfrak{b}\beta+\mathfrak{g}\gamma+\mathfrak{d}\delta)-t(-\mathfrak{b}\beta+\mathfrak{g}\gamma+\mathfrak{d}\delta)-t(\mathfrak{b}\beta-\mathfrak{g}\gamma+\mathfrak{d}\delta)-t(\mathfrak{b}\beta+\mathfrak{g}\gamma-\mathfrak{d}\delta))/24\\ &\quad=\mathfrak{b}\mathfrak{g}\mathfrak{d}(t(\beta+\gamma+\delta)-t(-\beta+\gamma+\delta)-t(\beta-\gamma+\delta)-t(\beta+\gamma-\delta))/24 \end{aligned}$$

for all ordered pairs β, γ, δ of real numbers, all real numbers $\mathfrak{b}$, $\mathfrak{g}$, $\mathfrak{d}$. The annihilator map $t(x,y)=0$ is a cubic form.

This paper gives the general characterization of polynomial transformations between vector spaces over a field of characteristic zero. The characterization, a generalization of A. M. Gleason's [3] and H. Röhrl's [9] recent treatment of quadratic forms, is in terms of inclusion-exclusion [4, pp. 8–10] identities. It is analogous to the characterization of a linear map v by means of the linearity identity $v(\mathfrak{a}\alpha+\mathfrak{b}\beta)=\mathfrak{a}v\alpha+\mathfrak{b}v\beta$. Constant, linear and affine maps do not fit

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neatly into the inclusion-exclusion identity theory. As far as it is concerned there is a disparity between the straight (degrees zero and one) and the curved (degrees two and higher).

2. **Euler homogeneous maps. Polynomial transformations.** Let V (with zero vector θ) and W (with zero vector ω) be vector spaces over a field $\mathfrak{K}$ of characteristic zero. Let J be the set W^V of functions (maps, transformations) with domain V, codomain W. Let r be a nonnegative integer. A map $s \in J$ is Euler homogeneous of degree r if for each $\mathfrak{a} \in \mathfrak{K}$, each $\alpha \in V$ it is true that $s\mathfrak{a}\alpha = \mathfrak{a}^r s\alpha$. A map $t \in J$ is a homogeneous polynomial transformation of degree r if there is an r-linear map $\boldsymbol{m}: V \times V \times \cdots \times V \rightarrow W$ such that for each $\alpha \in V$ it is true that $t\alpha = \boldsymbol{m}(\alpha, \alpha, \cdots, \alpha)$. Homogeneous polynomial transformation of degree r are Euler homogeneous maps of degree r. Let r be a nonnegative integer. A map $u \in J$ is a polynomial transformation of degree at most r if there is an r-affine map $\boldsymbol{a}: V \times V \times \cdots \times V \rightarrow W$ such that for each $\alpha \in V$ it is true that $u\alpha = \boldsymbol{a}(\alpha, \alpha, \cdots, \alpha)$. Let $t \in J$. If t is a homogeneous polynomial transformation of degree r then t is a polynomial transformation of degree at most r^* for each integer $r^* \geqq r$. Let $u \in J$. If u is an affine map then u is a polynomial transformation of degree at most r for each positive integer r. If u is a polynomial transformation of degree at most r then u is a polynomial transformation of degree at most r^* for each integer $r^* \geqq r$.

THEOREM. *Let $u \in J$. Let r be a nonnegative integer. Fix any integer $r^* \geqq r$. Suppose that u is a polynomial transformation of degree at most r^*. Suppose that u is an Euler homogeneous map of degree r. Then u is a homogeneous polynomial transformation of degree r.*

THEOREM. *Let r be a nonnegative integer. A map $u \in J$ is a polynomial transformation of degree at most r if and only if there is a homogeneous rth degree polynomial transformation $u^*: V \oplus \mathfrak{K} \rightarrow W$ such that for each $\alpha \in V$ it is true that $u\alpha = u^*(\alpha, 1)$.*

GLEASON'S LEMMA. *$t \in J$ is a homogeneous polynomial transformation of degree two if and only if both of the following assertions hold:*

(i) $$(t(\gamma + \mathfrak{b}\delta) - t(-\gamma + \mathfrak{b}\delta))/4 = \mathfrak{b}(t(\gamma + \delta) - t(-\gamma + \delta))/4$$

for each $\mathfrak{b} \in \mathfrak{K}$, each γ, $\delta \in V$.
(ii) *Either $t\mathfrak{a}\alpha = \mathfrak{a}^2 t\alpha$ for each $\mathfrak{a} \in K$, or else $t\theta = \omega$.*

3. **Inclusion-exclusion identities.** Now fix an integer $p \geqq 2$. Let $L = \{1, 2, \cdots, p\}$. If S is a finite set let $n(S)$ be the number elements of S. Thus $n(\phi) = 0$ and $n(L) = p$. Let

$$M = \{A \subset L \mid 2n(A) < p\} \cup \{A^* \subset L \mid 2n(A^*) = p \text{ and } 1 \in A^*\},$$
$$N = \{A \subset L \mid 1 \notin A\}.$$

Let 2^L be the power set of L. The set D^L consists of all lists [7, p. 43] of p elements of the set D. Define a function $r\colon 2^L \to \{1, -1\}^L$ by setting $r[B](j) = -1$ if $j \in B$, $r[B](k) = 1$ if $k \notin B$ for each $B \subset L$. Now let

$$\mathscr{S}(p, t, \alpha) = \sum_{A \in M} (-1)^{n(A)} t\left(\sum_{j \in L} r[A](j)\alpha(j)\right) \Big/ p!2^{p-1}$$

$$\mathscr{S}^*(p, u, \beta) = \sum_{B \in N} (-1)^{n(A)} u\left(\sum_{j \in L} r[B](j)\beta(j)\right).$$

If $\mathfrak{a} \in \mathfrak{R}^L$ and $\alpha \in V^L$ define the pointwise product $\mathfrak{a}\alpha \in V^L$ by setting $(\mathfrak{a}\alpha)(i) = \mathfrak{a}(i)\alpha(i)$ for each $i \in L$. Let

$$Y = \{\mathfrak{b} \in \mathfrak{R}^L \mid \mathfrak{b}(j) = 1 \quad \text{for each } j \in L \sim \{p\}\}.$$

A map $u \in J$ satisfies the inhomogeneous inclusion-exclusion identity of degree p if $\mathscr{S}^*(p, u, \mathfrak{b}\beta) = \mathfrak{b}(p)\mathscr{S}^*(p, u, \beta)$ for each $\mathfrak{b} \in Y$, each $\beta \in V^L$. A map $t \in J$ satisfies the heterogeneous inclusion-exclusion identity of degree p if $\mathscr{S}(p, t, \mathfrak{a}\alpha) = \mathfrak{a}(p)\mathscr{S}(p, t, \alpha)$ for each $\mathfrak{a} \in Y$, each $\alpha \in V^L$. A map $t \in J$ satisfies the homogeneous inclusion-exclusion identity of degree p if

$$\mathscr{S}(p, t, \mathfrak{a}\alpha) = \mathfrak{a}(1)\mathfrak{a}(2) \cdots \mathfrak{a}(p)\mathscr{S}(p, t, \alpha)$$

for each $\mathfrak{a} \in \mathfrak{R}^L$, each $\alpha \in V^L$.

THEOREM. *Let $u \in J$. Suppose u is an affine map. Then u satisfies the inhomogeneous inclusion-exclusion identity of degree p^* for each integer $p^* \geqq 2$.*

THEOREM. *Let $t \in J$. Suppose that t is an Euler homogeneous map of degree 1. Suppose that there is an integer $p^* \geqq 2$ such that t satisfies the inhomogeneous inclusion-exclusion identity of degree p^*. Then t is a linear map.*

THEOREM. *Let $t \in J$. Suppose that t is an Euler homogeneous map of degree p. Then t satisfies the homogeneous inclusion-exclusion identity of degree p if and only if t satisfies the heterogeneous inclusion-exclusion identity of degree p.*

To each $t \in J$ there corresponds a map $m[p, t]\colon V \times V \times \cdots \times V \to W$ defined by setting, for $\beta \in V^L$,

$$\boldsymbol{m}[p, t](\beta(1), \beta(2), \cdots, \beta(p)) = \mathfrak{s}(p, t, \beta).$$

THEOREM. *Let* $t \in J$. *Suppose* t *is an Euler homogeneous map of degree* p. *Suppose* t *satisfies the homogeneous inclusion-exclusion identity of degree* p. *Then* $\boldsymbol{m}[p, t]$ *is a symmetric multilinear map. Moreover for each* $\lambda \in V$ *it is true that* $t\lambda = \boldsymbol{m}[p, t](\lambda, \lambda, \cdots, \lambda)$.

This is a von Neumann-Jordan theorem [6] whose proof uses Gleason's lemma.

4. Intrinsic characterizations. Recall the blanket hypothesis of this paper, that $p \geqq 2$.

HOMOGENEOUS CHARACTERIZATION THEOREM. $t \in J$ *is a homogeneous polynomial transformation of degree* p *if and only if* t *is Euler homogeneous of degree* p *and* t *satisfies one of the following identities*:

(i) *The heterogeneous inclusion-exclusion identity of degree* p.

(ii) *The homogeneous inclusion-exclusion identity of degree* p.

(iii) *The inhomogeneous inclusion-exclusion identity of degree* p.

(iv) *The inhomogeneous inclusion-exclusion identity of some degree* $p^* \geqq p$.

(v) *The inhomogeneous inclusion-exclusion identity of each degree* $p^* \geqq p$.

INHOMOGENEOUS CHARACTERIZATION THEOREM. $u \in J$ *is a polynomial transformation of degree at most* p *if and only if* u *satisfies the inhomogeneous inclusion-exclusion identity of degree* p.

There are three ideas behind the proofs [1] of all these results. To prove that polynomial transformations satisfy inclusion-exclusion identities go back to the definitions in terms of multilinear and multiaffine maps. Write out the combinations and verify the identities. To prove that a degree p Euler homogeneous map $t \in J$, which satisfies that the homogeneous degree p inclusion-exclusion identity is in fact of the form

$$t\alpha = \boldsymbol{m}[p, t](\alpha, \alpha, \cdots, \alpha)$$

where $\boldsymbol{m}[p, t]$ is a symmetric multilinear map from $V \times V \times \cdots \times V$ to W, employ Gleason's Lemma and a combinatorial argument to show that $\boldsymbol{m}[p, t]$ is symmetric bilinear in any two of its arguments for a fixed setting of the other $p-2$. To get the general theory from the homogeneous theory without having to adapt all the proofs of the foregoing results to multiaffine maps employ the definition of arbitrary polynomial transformations from V to W in terms of homogeneous polynomial transformations from $V \oplus \mathfrak{R}$ to W. Some interesting technical lemmas are the following.

LEMMA. *Suppose* $u \in J$ *is a polynomial transformation of degree at most* p. *Define* t: $V \oplus \mathfrak{R} \to W$ *by setting*

$$t(\beta, 0) = \sum_{j=0}^{p} \binom{p}{j} (-1)^j u((p - 2j)\beta)/p!2^p$$
$$t(\beta, \mathfrak{b}) = \mathfrak{b}^p u((1/\mathfrak{b})\beta)$$

for each $\beta \in V$, *each nonzero* $\mathfrak{b} \in \mathfrak{R}$. *Then* t *is a homogeneous polynomial transformation of degree* p.

LEMMA. *Suppose* $t \in J$ *satisfies the homogeneous inclusion-exclusion identity of degree* p. *Then for each* $\beta \in V$ *it is true that* $(-1)^p t(-\beta) = t\beta$.

5. **Differential calculus.** Let V and W be Banach spaces. The Frechet derivative of t with respect to $\alpha \in V$ at β is the limit

$$\langle t: \alpha \rangle(\beta) = \lim(t(\beta + \mathfrak{h}\alpha) - t(\beta))/\mathfrak{h}$$

as $\mathfrak{h} \to 0$. The mixed pth order Frechet derivative [8, p. 169] of $t \in J$ with respect to the vectors on the list $\gamma \in V^L$ at the vector $\beta \in V$ is defined as

$$\langle t: \gamma(1), \gamma(2), \cdots, \gamma(p-1), \gamma(p) \rangle(\beta)$$
$$= \langle \langle t: \gamma(1), \gamma(2), \cdots, \gamma(p-1) \rangle : \gamma(p) \rangle(\beta).$$

The vector $\langle t: \delta, \delta, \cdots, \delta, \delta \rangle(\beta) = \langle t: \delta^p \rangle(\beta)$ is the pure pth Frechet derivative of t with respect to $\delta \in V$ at β.

EULER'S THEOREM. *If* $t \in J$ *is an Euler homogeneous map of degree* p *then for each* $x \in L$, *each* $\alpha \in V$ *it is true that*

$$(p - x)!\langle t: \alpha^x \rangle(\alpha) = p!t\alpha.$$

THE ARCHIMEDEAN MEAN VALUE THEOREM. *Suppose* $t \in J$ *is a homogeneous polynomial transformation of degree* $p+1$. *Suppose* $\alpha^* \in V^{L \cup \{p+1\}}$, that $\alpha^*(p+1) = \eta$, and that α is the restriction of α^* to L. Then

$$(p + 1)!2^p \mathfrak{s}(p + 1, t, \alpha^*)$$
$$= 2 \sum_{A \in M} (-1)^{n(A)} \langle t: \eta \rangle \left(\sum_{j \in L} r[A](j)\alpha(j) \right).$$

Suppose that $p = 1$, so that t is quadratic. If W, V and $\mathfrak{R}$ are the real numbers then the graph of t is a parabola through the origin. The theorem then implies that $t(\alpha+1) - t(\alpha-1) = 2t'(\alpha)$. This last fact, in its geometric form, was known to the Greeks [5, p. 234] before Archimedes. An induction based on this theorem leads to

DIXON'S THEOREM ON DIFFERENCES AND DERIVATIVES. *Suppose V and W are Banach spaces and that $t \in J$ is a homogeneous polynomial transformation of degree p. Then for each $\alpha \in V^L$ it is true that*

$$p!\mathfrak{d}(p, t, \alpha) = \langle t: \alpha(1), \alpha(2), \cdots, \alpha(p-1)\rangle(\alpha(p)).$$

This observation [2] of R. D. Dixon puts the foregoing theory into a new light. The theory was developed as a purely combinatorial exercise. But he has given very different proofs, valid in Banach spaces, of several of the results above.

Added in proof. S. Kurepa's papers (Glasnik Mat.-Fiz. Astronom. Ser. II Društvo Mat. Fiz. Hrvatske **19**(1964), 23–26 and **20**(1965), 79–92) parallel [**3**] and [**9**]. Polynomial transformations between affine [**7**, p. 420] spaces A, B over a field $\mathfrak{K}$ of characteristic zero have an intrinsic characterization. $\mathfrak{s} \in B^A$ is a quadratic polynomial transformation if and only if

$$\mathfrak{s}(\mathfrak{a}\alpha + \mathfrak{b}\beta) = \mathfrak{s}(\mathfrak{b}\alpha + \mathfrak{a}\beta) + (\mathfrak{a} - \mathfrak{b})\mathfrak{s}(\alpha) + (\mathfrak{b} - \mathfrak{a})\mathfrak{s}(\beta)$$

for each α, $\beta \in \mathrm{A}$ each $\mathfrak{a}$, $\mathfrak{b} \in \mathfrak{K}$ such that $\mathfrak{a}+\mathfrak{b}=1$. The characterization of a cubic polynomial transformation $s \in B^A$, in a symmetric form which can be given an intrinsic affine meaning, is that

$$\begin{aligned}[\mathfrak{s}(\mathfrak{a}\alpha + \mathfrak{b}\beta) - \mathfrak{s}(\mathfrak{a}\beta + \mathfrak{b}\alpha)] - [\mathfrak{s}(\mathfrak{a}(\Omega\alpha) + \mathfrak{b}(\Omega\beta)) - \mathfrak{s}(\mathfrak{a}(\Omega\beta) + \mathfrak{b}(\Omega\alpha))] \\ = [(\mathfrak{a}\mathfrak{s}(\alpha) + \mathfrak{b}\mathfrak{s}(\beta)) - (\mathfrak{a}\mathfrak{s}(\beta) + \mathfrak{b}\mathfrak{s}(\alpha))] \\ - [\mathfrak{a}\mathfrak{s}(\Omega\alpha) + \mathfrak{b}\mathfrak{s}(\Omega\beta) - (\mathfrak{a}\mathfrak{s}(\Omega\beta) + \mathfrak{b}\mathfrak{s}(\Omega\alpha))]\end{aligned}$$

for each α, $\beta \in \mathrm{A}$, each translation Ω of A, each $\mathfrak{a}$, $\mathfrak{b} \in \mathfrak{K}$ such that $\mathfrak{a}+\mathfrak{b}=1$. The affine inclusion-exclusion identity characterizations of higher degree transformations will appear in [1].

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