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ON AN ADDITIVE DECOMPOSITION OF FUNCTIONS OF SEVERAL COMPLEX VARIABLES

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1. **Introduction.** Recent attempts (see [1] and the references in the same article) to extend the Wiener-Hopf technique for functions of a single complex variable to those of two or more complex variables have relied on a remark of Bochner's [2] that guarantees the required decomposition under suitable restrictions. Bochner's remark states that: *if* $f(z_1, \cdots, z_n)$, $z_j = x_j + iy_j$, *is analytic in a tube* T: $\gamma_i < x_i < \delta_i$, $y_i \in (-\infty, \infty)$, *and if* $\int_{-\infty}^{\infty} \cdots \int |f(z_1, \cdots, z_n)|^2 dy_1 \cdots dy_n$ *converges in* T, *then there exists in* T *a decomposition* $f = \sum_{i=1}^{2^n} f_i$, *where each* f_i *is analytic and bounded in an octant shaped tube* T_i *containing the interior of* T. *Moreover, such a decomposition is unique up to additive constants.* The uniqueness of the decomposition is not verified in [2] but reference is made to H. Bohr's [3] corresponding result for functions of a single complex variable.

It is here shown that the uniqueness statement is false. However, the adjunction of the additional hypothesis that the $f_i \to 0$ when any one of the $x_j \to \infty$, in the tubes T_i, restores the uniqueness of the decomposition and justifies the use of the result in [2].

2. **A counter-example.** In the decomposition $f = \sum_{i=1}^{2^n} f_i$, f_1 is analytic and bounded in the tube T_1: $x_i > \gamma_i$, $y_i \in (-\infty, \infty)$, $i = 1, 2, \cdots, n$, and f_2 is analytic and bounded in the tube T_2: $x_1 < \delta_1$, $x_j > \gamma_j$,

$y_1 \in (-\infty, \infty)$, $y_j \in (-\infty, \infty)$, $j=2, 3, \cdots, n$. Let $g(z_2, z_3, \cdots, z_n)$ be any function of the $(n-1)$ complex variables $z_2, z_3, \cdots, z_n$ such that it is analytic and bounded for $x_j > \gamma_j$, $y_j \in (-\infty, \infty)$, $j=2, 3, \cdots, n$. In particular $\prod_{j=2}^{n} (z_j - \gamma_j + \epsilon)^{-1}$, $\epsilon > 0$, is such a function. Then the decomposition $f = \sum_{i=1}^{2^n} f_i'$, where $f_1' = f_1 + g$, $f_2' = f_2 - g$, $f_i' = f_i$, $i = 3, 4, \cdots, 2^n$, satisfies the conditions of Bochner's remark and yet the f_i and f_i' do not simply differ by a constant.

3. **Uniqueness of the decomposition.** The decomposition implied in Bochner's theorem is obtainable by the use of Cauchy integrals as in the case of functions of a single complex variable [**3**]. However, the f_i obtained from the Cauchy integrals when f is of bounded L_2 norm in T are not only bounded in T_i, but possess the asymptotic property $f_i \to 0$ as $x_j \to \pm \infty$ in the tube T_i, for any $j = 1, 2, \cdots, n$. It is just this asymptotic property that ensures the uniqueness of the decomposition.

THEOREM 1. *Let $f(z_1, \cdots, z_n)$ be analytic in T, and suppose that, in T, $f = \sum_{i=1}^{2^n} f_i$, where each of the f_i is analytic and bounded in T_i and $f_i \to 0$ as any one of the $x_j \to \pm \infty$ in T_i. Then the decomposition is unique.*

PROOF. Suppose that $f = \sum_{i=1}^{2^n} f_i = \sum_{i=1}^{2^n} f_i'$ in T. Then $\sum_{i=1}^{2^n} \Delta f^i = \sum_{i=1}^{2^n} (f_i - f_i') = 0$ in T. When $\gamma_j < x_j < \delta_j$, $y_j \in (-\infty, \infty)$, $j = 2, 3, \cdots, n$, 2^{n-1} of the terms in this last sum are analytic and bounded for $x_1 > \gamma_1$, $y_1 \in (-\infty, \infty)$, and $\to 0$ as $x_1 \to +\infty$, while the other 2^{n-1} terms are analytic for $x_1 < \delta_1$, $y_1 \in (-\infty, \infty)$, and $\to 0$ as $x_1 \to -\infty$. Denoting the sum of the first set by $[\Sigma \Delta f]_+$ and of the second by $[\Sigma \Delta f]_-$, it follows that $[\Sigma \Delta f]_+ = -[\Sigma \Delta f]_- = g(z_1, \cdots, z_n)$ in T. Now $g(z_1, \cdots, z_n)$ is analytic in all variables in T, and analytic and bounded for all z_1 whenever $\gamma_j < x_j < \delta_j$, $y_j \in (-\infty, \infty)$, $j = 2, 3, \cdots, n$. By Liouville's theorem $g(z_1, \cdots, z_n)$ is independent of z_1, say $g(z_1, \cdots, z_n) = G(z_2, z_3, \cdots, z_n)$. But $\lim_{x_1 \to \infty} [\Sigma \Delta f]_+ = 0$, hence $\lim_{x_1 \to \infty} G(z_2, \cdots, z_n) = G(z_2, \cdots, z_n) = 0$, and $[\Sigma \Delta f]_+ = 0$, $[\Sigma \Delta f]_- = 0$, in T. Now the same argument is applied to each of the above two equations on the variable z_2, resulting in four new equations of the same form, each involving 2^{n-2} summands. Repeating this process for $z_3, z_4, \cdots, z_n$, one finds that at each step the number of homogeneous equations is doubled, while the number of summands is halved. By the nth step, there are 2^n equations each involving one Δf_i. Thus $\Delta f_i = 0$ in T, and hence in T_i, $i = 1, 2, \cdots, n$, and the proof is complete.

COROLLARY 1. *Under the conditions of Bochner's theorem, $\exists f_i$ such* *that* $f=\sum_{i=1}^{2^n} f_i$ *in* T, *where the* f_i *are analytic and bounded in* T_i, *and* $f_i \to 0$ as $z_j \to \infty$, *for any* $j=1, 2, \cdots, n$, *in* T_i. *This decomposition is unique.*

COROLLARY 2. *If* $f(z_1, \cdots, z_n)$ *is analytic and bounded in* T, *and if* f *possesses a bounded indefinite integral* F *in* T *such that* $f=\partial^n F/\partial z_1 \cdots \partial z_n$, *then in* T, $f=\sum_{i=1}^{2^n} f_i$, *where* f_i *is analytic and bounded in* T_i, *and* $f_i \to 0$ *as* $x_j \to \pm\infty$, *for any* $j=1, 2, \cdots, n$, *in* T_i. *This decomposition is unique.*

PROOF. The existence of the decompositions postulated in the corollaries follows from the Cauchy integral theorem. The uniqueness is a consequence of Theorem 1.

The conditions imposed on f in Corollary 2 are direct extensions of the conditions that H. Bohr [3] imposed on functions of a single complex variable.

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