QUANTIZATION AND REPRESENTATIONS OF SOLVABLE LIE GROUPS

BY L. AUSLANDER AND B. KOSTANT[1]

Communicated by Felix Browder, April 10, 1967

Introduction. In this note, we will announce a characterization of a connected, simply connected Type I solvable Lie group, G, and present a complete description of the set of all unitary equivalence classes of irreducible unitary representations of G together with a construction of an irreducible representation in each equivalence class. This result subsumes the results previously obtained on nilpotent Lie groups and solvable Lie groups of exponential type of Kirillov [3] and Bernat [2], respectively.

Our result is made possible by a merging of a new general geometric approach to representation theory, based on the use of symplectic manifolds and quantization, of the second author with a detailed analysis of the Mackey inductive procedure which augments the results in [1].

1. **Outline of results.** Let (X, ω) be a symplectic manifold; i.e., a $2n$-dimensional manifold with a closed 2-form ω such that ω^n does not vanish on X and $d\omega=0$ on X. Let $[\omega]\in H^2(X, R)$ be the corresponding deRham class. A vital example of a symplectic manifold, for our purposes, is obtained as follows: Let G be a Lie group with Lie algebra $\mathfrak{g}$ and let $\mathfrak{g}'$ be the dual vector space to $\mathfrak{g}$. Then G acts on $\mathfrak{g}'$ by the contragredient representation and we will denote a G-orbit by O and the set of G-orbits by $\mathfrak{O}$. After several identifications it is possible to use the bilinear form $\langle f, [x, y]\rangle$, $x, y\in\mathfrak{g}, f\in\mathfrak{g}'$ to define a 2-form ω_O on each O such that (O, ω_O) is a symplectic manifold.

THEOREM 1. *Let G be a connected, simply connected solvable Lie group. Then G is Type* I *if and only if*

(a) *all G-orbits in $\mathfrak{g}'$ are G_δ sets in the usual topology on $\mathfrak{g}$.*

(b) $[\omega_O]=0$ *for all* $O\in\mathfrak{O}$.

REMARK. All algebraic Lie groups are Type I.

In general if (X, ω) is any symplectic manifold then there exists a complex line bundle L with connection α such that ω is the curvature form of the connection α, $\omega=\text{curv}(L, \alpha)$, if and only if the deRham

[1] The first author received partial support from the N.S.F. and U. S. Naval Research Laboratory and the second author received partial support from the N.S.F. and the Sloan Foundation.

cohomology class $[\omega]$ is integral. From the general theory one can show that if $\mathcal{H}_\omega$ denotes the set of all equivalence classes of pairs (L, α) such that

(1) $\omega = \operatorname{curv}(L, \alpha)$,

(2) there exists a parallel invariant Hermitian structure on L, then the character group, $\Pi_1^{\wedge}(X)$, of the fundamental group, $\Pi_1(X)$, of X operates in a simply transitive way on $\mathcal{H}_\omega$. In particular, $\mathcal{H}_\omega$ is not empty and, if the first betti number of X, b_1, is finite, $\mathcal{H}_\omega$ has the structure of a torus of dimension b_1. Note if $b_1=0$, then $\mathcal{H}_\omega$ has exactly one element.

If G is a connected Lie group we will write $\mathcal{H}_O$ for $\mathcal{H}_{\omega_O}$, where O contained in $\mathfrak{g}'$ is any G-orbit with integral class ω_O. Note by Theorem 1, if G is Type I, connected, simply connected solvable Lie group then ω_O is always integral.

If G is as in Theorem 1, then the general quantization procedure applied to this case enables us to naturally associate to each $\lambda \in \mathcal{H}_0$ an irreducible unitary representation Π_λ, described more explicitly in §2. Further, if $\lambda_i \in \mathcal{H}_{O_i}$, $i=1, 2$ then $\Pi_{\lambda_1} \simeq \Pi_{\lambda_2}$ if and only if $O_1 = O_2$ and $\lambda_1 = \lambda_2$. We will now denote the unitary equivalence class of Π_λ by λ and let $\hat{G}$ denote the set of all unitary equivalence classes of irreducible representations of G.

THEOREM 2. *If G is a connected, simply connected Type* I *solvable Lie group then $\hat{G} = \bigcup_{O \in \mathcal{O}} \mathcal{H}_O$ where $\mathcal{O}$ denotes the set of all G-orbits in $\mathfrak{g}'$.*

As noted above, H_O, has the structure of a torus of dimension $b_1(O)$, the first betti number of O. In case G is nilpotent, or, more generally, solvable of exponential Type, $\mathcal{H}_O$ has only one element and so $\hat{G} = \mathcal{O}$ and thus we obtain the results of Kirillov and Bernat.

2. **Description of $\prod_\lambda$.** Let $f \in O$ and let G_f be the isotropy group of f with Lie algebra $\mathfrak{g}_f$. One knows that $\mathfrak{g}/\mathfrak{g}_f$ has even dimension, say $2n$. If $\mathfrak{g}_C$ denotes the complexification of $\mathfrak{g}$, then a complex subalgebra $\mathfrak{h}$ of $\mathfrak{g}_C$ is called an *admissible* polarization of O (a concept from the general theory) in case

(1) $(\mathfrak{g}_f)_C \subseteq \mathfrak{h}$.

(2) $\mathfrak{h}$ is normalized by G_f. (Recall that, unlike the nilpotent or exponential case, G_f is not necessarily connected.)

(3) the following 5 conditions are satisfied:

(a) $\dim_C \mathfrak{g}_C/\mathfrak{h} = n$ (i.e., h is "half-way" between $(\mathfrak{g}_f)_C$ and $\mathfrak{g}_C$).

(b) f, regarded as an element of $\mathfrak{g}'_C$, vanishes on $[\mathfrak{h}, \mathfrak{h}]$.

(c) $\mathfrak{h}+\bar{\mathfrak{h}}$ is a subalgebra of $\mathfrak{g}_C$, where the bar denotes conjugation relative to $\mathfrak{g}$.

Let $\mathfrak{d}=\mathfrak{h}\cap\mathfrak{g}$ and let $\mathfrak{e}=(\mathfrak{h}+\bar{\mathfrak{h}})\cap\mathfrak{g}$. Hence $\mathfrak{d}\subseteq\mathfrak{e}$. Let ω_f be the alternating bilinear form on $\mathfrak{g}$ defined by $\omega_f(x, y)=\langle f, [y, x]\rangle$. It is clear that $\mathfrak{e}$ is the ω_f-orthogonal subalgebra to $\mathfrak{d}$, so that ω_f induces a nonsingular alternating bilinear from β_f on $\mathfrak{e}/\mathfrak{d}$ which implies that $\mathfrak{e}/\mathfrak{d}$ has even dimension.

Let j ($j^2=-1$) be the complex structure on $\mathfrak{e}/\mathfrak{d}$ induced by $\mathfrak{h}$. (The complexification of $\mathfrak{e}/\mathfrak{d}$ is $(\mathfrak{h}+\bar{\mathfrak{h}})/(\mathfrak{h}\cap\bar{\mathfrak{h}})$. This defines a nonsingular bilinear form σ_f on $\mathfrak{e}/\mathfrak{d}$ given by $\sigma_f(u, v)=\beta_f(u, jv)$.

(d) σ_f is positive definite.

In the terms of the general theory condition (d) states that a certain polarized symplectic manifold is Kahler. The final condition is the only one with no analogue in the general theory. We must add this requirement at this time, because there is as yet for the general class of solvable Lie groups no description of the *full* set of "good" polarizations of the kind obtained by Pukanszky [4] for the exponential case.

Let $\mathfrak{n}$ be the nil-radical of $\mathfrak{g}$, let $g\in n'$ be such that $g=f|\,\mathfrak{n}$ and let $\mathfrak{n}_g$ be the Lie algebra of the isotropy group of N, the Lie group with Lie algebra $\mathfrak{n}$, acting on $\mathfrak{n}'$. Then we have the dimension of $\mathfrak{n}/\mathfrak{n}_g$ is even, say $2m$. The final condition is that $\mathfrak{h}$ defines a polarization for the N-orbit on $\mathfrak{n}'$ determined by g. In view of the above discussion this means

(e) $\dim_C \mathfrak{h}\cap\mathfrak{n}_C=m$.

Now assume that $\mathfrak{h}$ is an admissible polarization. Then the product $G_fD=A$ is a subgroup of G, where D is the subgroup of G with Lie algebra $\mathfrak{d}=\mathfrak{h}\cap\mathfrak{g}$. This D is normalized by G_f. Further, the choice of $\lambda\in\mathcal{H}_O$ defines a character χ_λ of A. Let η_λ be the unitary representation of G induced by χ_λ and let J_λ be the corresponding Hilbert space. One may then regard the elements of J_λ as sections of a line bundle over G/A.

Now $B=G_fE$ is also a subgroup of G, where E is the subgroup of G with Lie algebra $\mathfrak{e}=\mathfrak{h}+\bar{\mathfrak{h}}$. It follows then that B/A has the structure of the previously mentioned Kähler manifold and its G transforms define a fibration of G/A by Kähler manifolds. Moreover the restriction of the line bundle to these fibers are holomorphic. Now let H_λ be the closed subspace of J_λ consisting of all sections in J_λ which are holomorphic along each fiber. Then H_λ is stable under $\eta_\lambda(G)$ and thus we obtain a representation Π_λ^h of G on H_λ. The irreducible unitary representation $\Pi_\lambda(G)$ associated with $\lambda\in\mathcal{H}_{\mathcal{O}}$ is the one given by the following theorem.

THEOREM 3. *Let G be a Type* I *connected, simply connected solvable Lie group. Then there exists an admissible polarization for any $O\in\mathcal{O}$;*

the unitary representation Π_λ^h *associated with h and* λ *is irreducible; and, up to unitary equivalence,* $\Pi\lambda_\lambda^h$ *is independent of the admissible polarizations.*

References

1. L. Auslander and C. C. Moore, *Unitary representations of solvable Lie groups*, Mem. Amer. Math. Soc. No. 62 (1966).

2. M. P. Bernat, *Sur les representations unitaires des groupes de Lie resolubles*, Ann. Sci. Ecole. Norm. Sup. **82** (1965), 37–99.

3. A. A. Kirillov, *Unitary representations of nilpotent Lie groups*, Uspehi. Mat. Nauk. **17** (1962), 57–110.

4. L. Pukanszky, *On the theory of exponential groups*, Trans. Amer Math. Soc. **126** (1967), 487–507.

City University of New York and
Massachusetts Institute of Technology