SOME GROUPS THAT ARE JUST ABOUT FREE

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Introduction. I have recently come across a rather extraordinary class of groups $\mathcal{G}$ while looking for a group of cohomological dimension 1 which is not free. The groups $G \in \mathcal{G}$ are generated by three elements a, b, c satisfying the single defining relation $a = c^{-i}a^{-1}c^{i}a . c^{-j}b^{-1}c^{j}b$:

$$G = (a, b, c; a = c^{-i}a^{-1}c^{i}ac^{-j}b^{-1}c^{j}b) \qquad (ij \neq 0). \tag{1}$$

The purpose of this announcement is to make known the surprisingly similar behavior of these groups in $\mathcal{G}$ and the free group F of rank two.

THEOREM. *Every group G in $\mathcal{G}$ satisfies the following conditions:*

(i) *G (like F) is the third term of an exact sequence*

$$1 \to N \to G \to Z \to 1$$

where N is free and Z is infinite cyclic;

(ii) *the 2-generator subgroups of G (like those of F) are free;*

(iii) *G (like F) is residually nilpotent, i.e.*

$$\bigcap_{i=1}^{\infty} \gamma_i G = 1,$$

where $\gamma_i G$ is the ith term of the lower central series of G;

(iv) *$G/\gamma_i G \cong F/\gamma_i F$ for $i = 1, 2, \cdots$;*

(v) *$G/G'' \cong F/F''$ where X'' is the second derived group of the group X;*

(vi) *G is not free.*[2]

Before making a few remarks about the proof of the Theorem I would like to point out that, by a theorem in [1], G is of cohomological dimension at most 2. Whether every group in $\mathcal{G}$ is of cohomological dimension precisely 2, I do not as yet know!

Incidentally, groups satisfying (iii) and (iv) (termed parafree in [2]) are plentiful [2]. The main point of the theorem is that nonfree parafree groups G satisfying (v) can exist.

Remarks on the proof of the theorem. Let G be the group given by (1). We verify that G has the properties (i)–(vi).

[1] The author is a Sloan Fellow.

[2] I thank S. Meskin for helping to verify (vi).

(i) is straightforward and is proved by taking N to be the normal closure of a and b and applying the Reidemeister-Schreier procedure for finding generators and relations for a subgroup of a group given by generators and relations (see e.g. [3, p. 86]).

(ii) holds for all parafree groups [4].

(iii) is the most difficult property to verify. One proves that G has this property by making use of the exact sequence (i).

(iv) and (v) follow exactly from the observation that G is the freest group generated by a, b and c satisfying the relation $a^{-1}c^{-i}a^{-1}c^{i}ac^{-j}b^{-1}c^{j}b=1$.

(vi) There is an algorithm introduced by J. H. C. Whitehead [5] whereby one can effectively determine whether a group with a single defining relation is free. (vi) follows on applying this algorithm to G.

References

1. R. C. Lyndon, *Cohomology theory of groups with a single defining relation*, Ann. of Math. (2) **52** (1950), 650–665.

2. G. Baumslag, *Groups with the same lower central sequence as a relatively free group*. I: *The groups*, Trans. Amer. Math. Soc. (to appear).

3. W. Magnus, A. Karrass and D. Solitar, *Combinatorial group theory*, Interscience, New York, 1966.

4. G. Baumslag, *Groups with the same lower central sequence as a relatively free group*. II: *Properties*, Trans. Amer. Math. Soc. (to appear).

5. J. H. C. Whitehead, *On certain sets of elements in a free group*, Proc. London Math. Soc. **41** (1936), 48–56.

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