SOME RESULTS ON LIE p-ALGEBRAS

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Let $\mathfrak{L}$ be a Lie p-algebra ("restricted Lie algebra") over the field $\mathfrak{F}$ of prime characteristic p [3, Chapter V]. Denote by x^p the image of $x \in L$ under the p-power operation, by x^{p^k} the image of x under the kth iterate of $x \rightarrow x^p$, with $x^{p^0} = x$. Let $\langle x \rangle$ be the subalgebra of $\mathfrak{L}$ generated by x, i.e., the space of linear combinations of the x^{p^k}, $k = 0, 1, 2, \cdots$. Call $x \in \mathfrak{L}$ *separable* if $x \in \langle x^p \rangle$, *nilpotent* if $x^{p^k} = 0$ for some k. Then we have proved the following decomposition theorem, which yields a slightly sharpened form of the Jordan-Chevalley decomposition [2, p. 71] for linear transformations in the case of prime chtracteristic.

THEOREM 1. *Let $x \in \mathfrak{L}$, a Lie p-algebra of finite dimension over the perfect field $\mathfrak{F}$. Then there exist elements s, $n \in \langle x \rangle$ with s separable and n nilpotent, such that $x = s + n$. If $y \in \mathfrak{L}$ is separable, $z \in \mathfrak{L}$ nilpotent, $[yz] = 0$, and $x = y + z$, then $y = s$ and $z = n$.*

A subalgebra $\mathfrak{T}$ of the Lie p-algebra $\mathfrak{L}$ is called *toral* if $\mathfrak{T}$ is commutative and if every element of $\mathfrak{T}$ is separable. A subalgebra $\mathfrak{N}$ is called *nil* if every element of $\mathfrak{N}$ is nilpotent. For a Lie p-algebra $\mathfrak{L}$ of endomorphisms of a finite-dimensional vector space over an algebraically closed field, to say that $\mathfrak{L}$ is triangulable is to say that $[\mathfrak{L}\mathfrak{L}]$ is nil. In this connection we have the following result.

THEOREM 2. *Let $\mathfrak{L}$ be a Lie p-algebra over the perfect field $\mathfrak{F}$, and suppose that $[\mathfrak{L}\mathfrak{L}]$ is nil. Let $\mathfrak{N}$ be the set of nilpotent elements of $\mathfrak{L}$, and let $\mathfrak{T}$ be any maximal toral subalgebra of $\mathfrak{L}$. Then $\mathfrak{N}$ is an ideal in $\mathfrak{L}$, and $\mathfrak{L} = \mathfrak{T} + \mathfrak{N}$. If, moreover, $\mathfrak{L}$ is nilpotent (as ordinary Lie algebra), then $\mathfrak{T}$ is the set of all separable elements of $\mathfrak{L}$ and $\mathfrak{T}$ is central in $\mathfrak{L}$.*

As to conjugacy of maximal toral subalgebras under these conditions we have shown the following:

THEOREM 3. *Let $\mathfrak{L}$ be a Lie p-algebra over the field $\mathfrak{F}$. Suppose that the set $\mathfrak{N}$ of nilpotent elements is an ideal in $\mathfrak{L}$, and let $\mathfrak{T}_1$ and $\mathfrak{T}_2$ be toral subalgebras such that $\mathfrak{T}_i + \mathfrak{N} = \mathfrak{L}$. If $\mathfrak{N}$ is commutative, then there is an automorphism σ of the Lie p-algebra $\mathfrak{L}$ such that $x^\sigma = x$ for all $x \in \mathfrak{N}$, with $y^\sigma - y \in \mathfrak{N}$ for all $y \in \mathfrak{L}$, and with $\mathfrak{T}_1^\sigma = \mathfrak{T}_2$. In general, there is no*

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automorphism of the Lie p-algebra $\mathfrak{L}$ mapping $\mathfrak{T}_1$ on $\mathfrak{T}_2$, even when $\mathfrak{F}$ is algebraically closed and $[\mathfrak{L}[\mathfrak{N}\mathfrak{N}]]=0$.

Over perfect fields in general, maximal toral subalgebras are related to Cartan subalgebras by the following:

THEOREM 4. *Let $\mathfrak{L}$ be a Lie p-algebra over a perfect field $\mathfrak{F}$. Let $\mathfrak{T}$ be a maximal toral subalgebra of $\mathfrak{L}$, $\mathfrak{H}=N(\mathfrak{T})$ the normalizer of $\mathfrak{T}$. Then $\mathfrak{H}$ is a Cartan subalgebra of $\mathfrak{L}$. Conversely, if $\mathfrak{H}$ is a Cartan subalgebra of $\mathfrak{L}$, then $\mathfrak{H}=N(\mathfrak{T})$, where $\mathfrak{T}$ is the set of separable elements of $\mathfrak{H}$, and $\mathfrak{T}$ is a maximal toral subalgebra of $\mathfrak{L}$.*

THEOREM 5. *Let $\mathfrak{L}$ be a Lie p-algebra over a perfect field $\mathfrak{F}$. Let (x, y) be a nondegenerate symmetric associative (i.e., $([xy], z)=(x, [yz])$) bilinear form on $\mathfrak{L}$ such that $(x, y)=0$ whenever y is nilpotent and $[xy]=0$. Then the Cartan subalgebras of $\mathfrak{L}$ are the maximal toral subalgebras.*

From Theorem 4 and the usual proof for infinite fields [**3**] we see that all Lie p-algebras possess Cartan subalgebras. The conditions of Theorem 5 are satisfied by trace forms of p-representations whenever such forms are nondegenerate, and also by the usual "quotient trace form" [**6**] on the pr by pr $\mathfrak{F}$-matrices of trace zero, modulo scalars.

Our further results concern toral algebras. If $\mathfrak{K}$ is any field of characteristic p then $\mathfrak{K}$, with its natural p-power, is a one-dimensional toral algebra over $\mathfrak{K}$, and may be regarded as a Lie p-algebra over any subfield of $\mathfrak{K}$. In this sense, we call a *$\mathfrak{K}$-valued character* of a toral algebra $\mathfrak{L}$ over $\mathfrak{F}$ an $\mathfrak{F}$-homomorphism of Lie p-algebras of $\mathfrak{L}$ into the extension $\mathfrak{K}$ of $\mathfrak{F}$. We then can prove

THEOREM 6. *Let $\mathfrak{L}$ be a (finite-dimensional) toral Lie p-algebra over $\mathfrak{F}$. Let $\mathfrak{K}$ be an extension field of $\mathfrak{F}$. Then the following are equivalent:*

(1) *All characters of $\mathfrak{L}$ with values in an extension of $\mathfrak{K}$ are $\mathfrak{K}$-valued;*

(2) *$\mathfrak{L}_{\mathfrak{K}}$ is isomorphic to a direct sum of copies of $\mathfrak{K}$.*

Such an extension $\mathfrak{K}$ will be called a *splitting field* for $\mathfrak{L}$; by appeal to (1) it is not hard to see that $\mathfrak{L}$ has a finite splitting field, and indeed a unique minimal one within a given algebraic closure of $\mathfrak{F}$. This field is a galois extension of $\mathfrak{F}$.

THEOREM 7. *Let $\mathfrak{L}$ be toral over $\mathfrak{F}$. Consider the properties:*

(a) *the only $\mathfrak{F}$-valued character of $\mathfrak{L}$ is zero;*

(b) *$\mathfrak{L}$ contains no subalgebra isomorphic to $\mathfrak{F}$.*

If $\mathfrak{F}$ is finite, (a) *and* (b) *are equivalent. On the other hand, there exist fields $\mathfrak{F}$ of all characteristics $p>2$ and two-dimensional toral algebras over $\mathfrak{F}$ which violate each of the implications* (a)$\Rightarrow$(b), (b)$\Rightarrow$(a). *In these examples the field $\mathfrak{F}$ may be taken to be perfect.*

A toral algebra $\mathfrak{L}$ over $\mathfrak{F}$ is called *anisotropic* if condition (a) of Theorem 7 is satisfied, *semisplit* if $\mathfrak{L}$ has a composition series with factors isomorphic to $\mathfrak{F}$. A semisplit toral algebra need not be split, even over a finite field.

THEOREM 8. *Let $\mathfrak{L}$ be toral over $\mathfrak{F}$. Then $\mathfrak{L}$ has a unique maximal anisotropic subalgebra $\mathfrak{A}$ and a unique maximal semisplit subalgebra $\mathfrak{S}$. If $\mathfrak{F}$ is finite, $\mathfrak{L}=\mathfrak{A}\oplus\mathfrak{S}$. There exist examples, as in Theorem 7, where $\mathfrak{L}=\mathfrak{A}+\mathfrak{S}$, $\mathfrak{A}\cap\mathfrak{S}\neq 0$, and examples where $\mathfrak{A}\cap\mathfrak{S}=0$ but $\mathfrak{L}\neq\mathfrak{A}+\mathfrak{S}$.*

The results above are imperfectly analogous with some for algebraic tori [1], [4], [5]. The final two theorems relate toral algebras and algebraic tori:

THEOREM 9. *Let T be an algebraic torus defined over the field $\mathfrak{F}$ of characteristic $p\neq 0$. Let $\mathfrak{K}$ be a minimal (separable) splitting field for T, $\mathfrak{G}$ the Galois group of $\mathfrak{K}/\mathfrak{F}$, $\mathfrak{L}=\mathfrak{L}(T)$ the Lie algebra of T. Then $X(\mathfrak{L})$, the character group (under addition) of $\mathfrak{L}$, is isomorphic with $X^*(T)/pX^*(T)$ as groups with $\mathfrak{G}$ as operators ($X^*(T)$ = character group of T). If $p\neq 2$, $\mathfrak{K}$ is a minimal splitting field for $\mathfrak{L}$, so that T is split if and only if $\mathfrak{L}(T)$ is. This assertion fails for $p=2$.*

If $m=\dim T=\dim \mathfrak{L}(T)$, then $X^*(T)$ is isomorphic to the free abelian group $\mathbf{Z}^m$; if $\mathfrak{L}$ is toral of dimension m, $X(\mathfrak{L})$ is an elementary p-group of order p^m. Thus the following is a converse to Theorem 9:

THEOREM 10. *Let $\mathfrak{L}$ be an m-dimensional toral algebra over $\mathfrak{F}$. Suppose there is a finite galois extension $\mathfrak{K}$ of $\mathfrak{F}$ splitting $\mathfrak{L}$, an action of $\mathfrak{G}=\mathfrak{G}(\mathfrak{K}/\mathfrak{F})$ on $\mathbf{Z}^m$, and a $\mathfrak{G}$-homomorphism of $\mathbf{Z}^m$ onto $X(\mathfrak{L})$. Then there is an algebraic torus T defined over $\mathfrak{F}$, split by $\mathfrak{K}$, such that $\mathfrak{L}$ is isomorphic to $\mathfrak{L}(T)$.*

Proofs of these results will appear elsewhere.

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