AN ISOMORPHISM PRINCIPLE IN GENERAL TOPOLOGY

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Introduction. To practically every topological space T of importance (including metrizable and locally compact Hausdorff spaces) one can let correspond (essentially by interchanging compact and closed sets) an "antispace" T^* which conversely determines T. If, for example, T is Hausdorff but not compact, T^* will be T_1, compact and superconnected (every open set is connected). One sacrifices the Hausdorff property but gains e.g. compactness. Furthermore the topology of T^* is weaker than that of T. This destroys the belief, generally held, that non-Hausdorff spaces are of minor or no importance. On the contrary, one could even say that they are "more elegant," since they perform the same job with a weaker topology.

Philosophically, the consequences seem to be of interest. If R denotes "time" (the real line), R^* has the same topology as R on every bounded closed interval. However R^* is compact. Time becomes unbounded but finite in the sense of compact. We have potential but no actual infinity.

A remark by J. M. Aarts (in our joint work on cocompactness) initiated this note.

Preliminaries. Let X be a set and $\{G\}$ a family of subsets G of X, closed under finite unions and arbitrary intersections. We do *not* assume the (usual) convention that X and $\varnothing$ are necessarily members of $\{G\}$. A pair $T_-=(X, \{G\})$ is called a (topological) *minusspace*, where $\{G\}$ indicates the family of closed sets of T_-. One can, of course, extend every T_- to a *topological* space T by adding X and $\varnothing$ as closed sets.

A subset S of T_- is called *squarecompact relative* to T_-, if for every family $\{C_\alpha\}$ of compact subsets C_α of T_-, for which $\{S\cap C_\alpha\}$ is centered (that is the intersection of finitely many $S\cap C_\alpha$ is nonempty), the intersection of all $S\cap C_\alpha$ is nonempty.

One can prove:

(i) The intersection of a compact and a squarecompact set is both compact and squarecompact.

(ii) The union of finitely many and the intersection of any number of squarecompact sets is squarecompact.

(iii) If in T_- every compact set is closed, then every closed set is squarecompact.

A topological space is called a *c-space* if the closed sets are exactly those sets for which the intersection with every compact closed set is compact. This notion is different from the well-known notion of a k-space (compactly generated space). However, every k-space is a c-space and for those spaces in which compact sets are closed, both notions coincide.

The following notions are equivalent for a topological space T:

(a) It is a c-space.

(b) Compact sets are closed and squarecompact sets are closed.

(c) Firstly, the intersection of a compact and a squarecompact set is closed; secondly, a set G is closed in T iff $G \cap C$ is closed for all compact sets C of T.

(d) Firstly, the intersection of two compact sets is compact; Secondly, G is closed, iff $G \cap C$ is compact for all compact C.

Which spaces are c-spaces?

(iv) In every locally compact topological space (that is, every point has arbitrarily small compact neighborhoods) or in every space satisfying the first axiom of countability the following properties are equivalent:

(a) Every compact set is closed,

(b) The space is a c-space,

(c) The space is Hausdorff.

Antispaces. Let X be a set, $T_- = (X, \{G\}, \{C\})$, $T_-^* = (X, \{C\}, \{G\})$ two minusspaces over X, where $\{G\}$ denotes the family of all closed sets G and $\{C\}$ the family of all compact sets C in T_-, while $\{C\}$ are the *closed* sets of T_-^* and $\{G\}$ the *compact* sets of T_-^*. So the identity map of X onto itself maps the closed (compact) sets of X onto the compact (closed) sets of T_-^*. Such a pair is called an antipair and T_- and T_-^* are called *antispaces* (of each other). A space T_- is an antispace, if there exists a T_-^* as indicated. Observe that antispaces determine each other. If in T_- the compact sets coincide with the closed sets, T_- and T_-^* coincide as e.g. is the case for compact Hausdorff spaces.

EXAMPLE. If X is any set, and T the discrete space over X, then T^* is determined by the cofinite topology on X.

THEOREM. *A minusspace is an antispace, iff the closed sets coincide with the squarecompact sets. The topological antispaces T are exactly the c-spaces.*

Observe that the *compact* anti(minus)spaces T^* pair off with the *topological* antispaces T. A minusspace is a compact antispace, iff every closed set is compact and the squarecompact sets coincide with the closed sets. Notice that according to (iv), most spaces of importance in mathematics are c-spaces.

A category of antispaces T and onto continuous mappings f corresponds to the category of spaces T^* and onto mappings f^* (f^* defined by the requirement, that the inverse of any compact image set is compact). This sets up an isomorphism as mentioned in the title.

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SOME EXPLICIT POLYNOMIAL APPROXIMATIONS IN THE COMPLEX DOMAIN

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1. Let $f(z)$ be continuous on a compact set, B, in the complex plane. Then if B contains more than n points $f(z)$ has a unique best uniform approximation out of the polynomials of degree n. That is, if P_n is the set of polynomials of degree at most n, there exists $p_n^* \in P_n$ satisfying

$$(1) \qquad \|f - \overset{*}{p}_n\| < \|f - p\|$$

for all $p \in P_n$, $p \neq p_n^*$, the norm being the uniform norm. It is an instance of a result due to Kolmogorov (see Meinardus [2; p. 15], for example) that $p_n^* \in P_n$ satisfies (1) if, and only if,

$$(2) \qquad \min_{z \in E} \operatorname{Re}[f(z) - \overset{*}{p}_n(z)]\bar{p}(\bar{z}) \leqq 0, \quad p \in P_n,$$

where

$$(3) \qquad E = \{z / |f(z) - \overset{*}{p}_n(z)| = \|f - \overset{*}{p}_n\|\}.$$

Let us put

$$(4) \qquad \rho_n(f; B) = \|f - \overset{*}{p}_n\|.$$

In [1] Al'per showed that

$$(5) \qquad \rho_n(z^s/(z^p - a^p); K_R) = R^{pk+p+s}/(|a|^{2p} - R^{2p})|a|^{pk}$$