# QUASI-PERIODIC SOLUTIONS OF NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS WITH SMALL DAMPING

BY MARCELLE FRIEDMAN

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A function $f(t)$ is called quasi-periodic if it can be represented in the form

$$f(t) = F(\omega_1 t, \omega_2 t, \cdots, \omega_m t)$$

where $F(\theta_1, \cdots, \theta_m)$ is a continuous function of period $2\pi$ in $\theta_\nu$, $\nu = 1, \cdots, m$. The numbers $\omega_1, \cdots, \omega_m$ are called the basic frequencies of $f(t)$. We shall denote by $A(\omega_1, \cdots, \omega_m)$ the class of all functions $f$ for which $F$ is real analytic. For simplicity of notation we set $\theta = (\theta_1, \cdots, \theta_m)$ and $\omega = (\omega_1, \cdots, \omega_m)$ (then $A(\omega_1, \cdots, \omega_m) = A(\omega)$ and $F(\theta_1, \cdots, \theta_m) = F(\theta)$).

The purpose of this note is to study the family of complex systems of differential equations:

$$\begin{aligned} \dot{z} &= \lambda z + \epsilon f(t, z, \bar{z}), \\ \dot{\theta} &= \omega \end{aligned} \tag{1}$$

parametrized by $\lambda$, $f$ analytic in $z$, $\bar{z}$, and $f \in A(\omega)$—i.e. $f(t, z, \bar{z}) = g(\theta, z, \bar{z})$ where $g$ is $2\pi$-periodic in $\theta$—to determine the complex numbers, $\lambda$, for which there exists a solution $z = \phi(t, \epsilon) \in A(\omega)$.[1]

For $\operatorname{Re} \lambda = 0$ there may be no solutions even in the linear case

$$\begin{aligned} \dot{z} &= \lambda z + \epsilon g(\theta), \\ \dot{\theta} &= \omega \end{aligned} \tag{2}$$

because of resonance. It is well known that if $\operatorname{Re} \lambda \neq 0$ and $\epsilon > 0$ is small compared with $|\operatorname{Re} \lambda|$ then (1) always has a solution $z = \phi(t, \epsilon) \in A(\omega)$. This was shown by Malkin [7] and Bohr and Neugebauer [4] in the linear case and by Stoker [10] and, in the general case, by Bogoliubov [1].

Our main interest is $|\operatorname{Re} \lambda|$ small compared to $\epsilon$. We shall describe a domain, $\Omega$, in the $\lambda$-plane such that for each $\lambda \in \Omega$ the corresponding system (1) has a solution $z = \phi(t, \epsilon) \in A(\omega)$. We call $\Omega$ a nonresonance domain. We will show that $\Omega$ contains in particular $|\operatorname{Re} \lambda| > 1$ (this

[1] This system is derived from the second order equation $\ddot{x} + c\dot{x} + ax = f(t, x, \dot{x})$ ($f$ quasi-periodic in $t$) by the transformation $z = \dot{x} + \alpha x$ for some constant $\alpha$.

corresponds to the above-mentioned result of Bogoliubov) and in the remaining strip consists of a collection of closed sets each connecting the two half planes which we will call $\Omega_+$ and $\Omega_-$. (See Figure 1.) Moreover, the complement of these closed sets has small measure, independent of $\epsilon$.

We set $(\omega, k) = \sum_{\nu=1}^{m} \omega_\nu k_\nu$ where the $k_\nu$, $\nu = 1, \cdots, m$ are integers, and $|k| = \sum_{\nu=1}^{m} |k_\nu|$. If we assume that $g$ is analytic for $|z|$, $|\bar{z}| < r$, $|\text{Im}\,\theta| = \sum |\text{Im}\,\theta_\nu| < 1$ and that $|g| < 1$, then

THEOREM. *If* $|(\omega, k)| \geqq c_0^{-1}|k|^{-\tau}$, $c_0 > 1$, $\tau > m$, *then there exists* $\epsilon_0 = \epsilon_0(m)$ *such that for* $\epsilon \leqq \epsilon_0$ *there exists a closed, connected set,* $\Omega = \Omega(\epsilon)$ *in the* $\lambda$*-plane such that for the corresponding system*

$$(1)' \qquad \begin{aligned} \dot{z} &= \lambda z + \epsilon g(\theta, z, \bar{z}), \\ \dot{\theta} &= \omega \end{aligned}$$

*has a solution*

$$z = \phi(t, \epsilon) \in A(\omega).$$

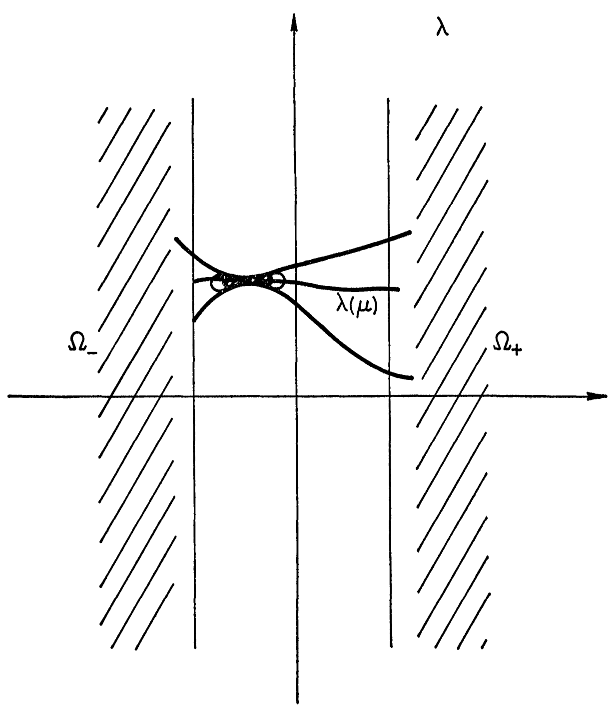


FIGURE 1

The set $\Omega$ contains the half planes $\Omega_+$ and $\Omega_-$. The latter two sets are connected by infinitely many cusp-like domains bounded by curves with one point of contact. The quasi-periodic solutions are stable or unstable according as $\lambda$ lies to the left or right of the contact point.

It should be noted here that although $\Omega$ depends on $\epsilon$ and $g$ the measure of the complement is small independent of the perturbation. This implies that for most choices of $\lambda$ the system (1)′ has a solution $\in A(\omega)$.

The complement of $\Omega$ is not empty even in the linear case. This can be seen as follows:

We find a solution $z$ of the linear problem by means of Fourier series. Substituting in $\dot{z} = \lambda z + \epsilon g(\theta)$ we obtain the following equations for the Fourier coefficients $z_k$ of $z$:

$$\{i(\omega, k) - \lambda\} z_k = \epsilon g_k.$$

For Re $\lambda = 0$, $|i(\omega, k) - \lambda|$ can be arbitrarily small since $\omega_1, \cdots, \omega_m$ are rationally independent. To prevent this we must restrict the choice of $\lambda$. If we require that $\lambda$ satisfy the inequality $|i(\omega, k) - \lambda| \geqq \gamma^{-1}$ for some constant $\gamma > 1$, we find that all pure imaginary $\lambda$ are excluded. However, if we weaken the condition to

$$|i(\omega, k) - \lambda| \geqq (\gamma |k|^{\tau})^{-1}$$

where $\gamma > 1$, $\tau > m$, we find that the measure of the excluded set on any line parallel to the imaginary axis is proportional to $\gamma^{-1}$ and decreases as $|\text{Re } \lambda|$ increases. Hence there are pure imaginary $\lambda$ for which $\dot{z} = \lambda z + \epsilon g(\theta)$ has a formal solution $z$ (convergence is assured if $g(\theta)$ is sufficiently differentiable).

The proof of our theorem is divided into two steps. The first and main step will consist of finding a family of curves, $\Gamma$, in the $\lambda$-plane such that for the corresponding differential equation we can

(i) construct quasi-periodic solutions belonging to $A(\omega)$,

(ii) transform the linearized equation (linearized on these solutions) into constant coefficients.

If $|\text{Re } \lambda| > 1$ we can easily use the contraction principle on the iteration scheme

$$z_0 = 0,$$

$$\dot{z}_{n+1} - \lambda z_{n+1} = \epsilon g(\theta, z_n, \bar{z}_n)$$

and show convergence for $\epsilon / |\text{Re } \lambda|$ sufficiently small. (This is essentially the technique of Bogoliubov [2], Stoker [10], and Malkin [7].) Our main interest, however, is for $|\mathrm{Re}\,\lambda|$ small. Here we need the "rapid convergence" technique of Kolmogorov [5], [6], Arnol'd [1], and Moser [8]. More precisely, we proceed as follows.

We construct a quasi-periodic transformation $z=\zeta+v(\theta, \zeta, \bar{\zeta}, \lambda)$ taking (1)′ into

$$\dot{\zeta} = \mu\zeta + \phi(\theta, \zeta, \bar{\zeta}, \mu) = \mu\zeta + \theta(|\zeta|^2),$$
$$\dot{\theta} = \omega$$

where $\mu$ satisfies

$$|(\omega, k) - j_0 \,\mathrm{Im}\,\mu| \geqq (\gamma |k|^\tau)^{-1},$$
$$\gamma > 1, \quad \tau > m, \quad |k| \neq 0, \quad j_0 = 0, 1, 2.$$

This provides a quasi-periodic solution $z=v(\omega t, 0, 0, \lambda)\in A(\omega)$ on a nondenumerable set of curves connecting $\Omega_+$ and $\Omega_-$.

In the second step of the proof, in order to enlarge the domain we must give up the requirement that the linearized equation be transformable into constant coefficients. For every $\mu$ with $\mathrm{Re}\,\mu\neq 0$ using a contraction argument we can ensure the existence of a solution $z\in A(\omega)$ if $\lambda$ is sufficiently close to the above determined curves, $\lambda=\lambda(\mu)$. It suffices to take $|\lambda-\lambda(\mu)| < c|\mathrm{Re}\,\mu|^2$. This determines for each curve in $\Gamma$ a parabolic neighborhood (see Figure 1) with point of contact at $\mathrm{Re}\,\mu=0$.

It should be noted here that the point of contact need not be on $\mathrm{Re}\,\lambda=0$. However, for reversible systems $(g(\theta, z, \bar{z}) = [-\bar{g}(-\theta, -\bar{z}, -z)])$ it was shown by Moser [9] that all contact points lie on $\mathrm{Re}\,\lambda=0$.

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SETON HALL UNIVERSITY