SPLINE FUNCTION APPROXIMATIONS FOR SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS

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1. Introduction and description of method. The known discrete variable methods for the solutions of differential equations (see [1]) furnish the approximate solutions as discrete tabular values at usually equidistant values of the independent variable. The object of this paper is to search for approximate solutions in the form of a spline function $S(x)$, [4], of degree m $(m \geq 2)$ and class C^{m-1}. This approach was suggested to the authors by I. J. Schoenberg [3].

Let the differential equation be

$$(1) \qquad y' = f(x, y), \qquad 0 \leq x \leq b,$$

about which we assume the following. If $T = \{(x, y) \mid 0 \leq x \leq b\}$, then we assume that $f(x, y) \in C^{m-2}$ in T and that it satisfies the Lipschitz condition

$$(2) \qquad |f(x, y) - f(x, y^*)| \leq L |y - y^*| \qquad \text{if } 0 \leq x \leq b.$$

If $m \geq 3$ then (2) is equivalent to the boundedness of $\partial f / \partial y$ in T.

Our construction of the approximate solution $S(x) = S_m(x)$ is as follows. Let $y(x)$ be the solution of (1) determined by the initial value $y(0) = y_0$. Let n be an integer $> m$, $h = b/n$ and let $S(x)$ $(0 \leq x \leq b)$ be a spline function of degree m, class C^{m-1} and having its knots at the points $x = h, 2h, \cdots, (n-1)h$.

We define the first component of $S(x) = S_m(x)$ by

$$
\begin{aligned}
S(x) = y(0) + y'(0)x + \cdots + & \frac{1}{(m-1)!} y^{(m-1)}(0) x^{m-1} \\
(3) & \\
& + \frac{1}{m!} a_0 x^m \qquad (0 \leq x \leq h)
\end{aligned}
$$

with the last coefficient a_0 as yet undetermined. We now determine a_0 by requiring that $S(x)$ should satisfy (1) for $x = h$. This gives the equation

$$(4) \qquad S'(h) = f(h, S(h))$$

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to be solved for a_0. In terms of $\xi = a_0 h^{m-1}(1-m)!$ it is seen that (4) is an equation in ξ which is conveniently solved by iteration.

The polynomial (3) being determined, repeat the same steps in the interval $[h, 2h]$: we define

$$(5) \quad S(x) = \sum_{\nu=0}^{m-1} \frac{1}{\nu!} S^{(\nu)}(h)(x-h)^\nu + \frac{1}{m!} a_1(x-h)^m \quad (h \leq x \leq 2h)$$

and determine a_1 such as to satisfy the equation

$$S'(2h) = f(2h, S(2h)).$$

Continuing in like manner we evidently obtain a spline function $S_m(x)$ satisfying the equation

$$(6) \qquad S_m'(\nu h) = f(\nu h, S_m(\nu h)) \qquad (\nu = 0, 1, \cdots, n).$$

THEOREM 1. *If $h < m/L$ then the spline function $S_m(x)$ exists and is uniquely defined by the above construction.*

2. **The consistency relation for a spline function.** Let $\mathcal{S}$ denote the class of spline functions in $[0, b]$, of degree m, class C^{m-1} with knots $x_\nu = \nu h$ $(\nu = 1, \cdots, n-1)$. Let $s(x) \in \mathcal{S}$. *If restricted to the interval* $[0, (m-1)h]$, $s(x)$ depends on $(m+1)+(m-2) = 2m-1$ linear parameters. It follows that the $2m$ quantities,

$$s(\nu h), s'(\nu h) \qquad (\nu = 0, 1, \cdots, m-1),$$

cannot be linearly independent. In fact there is a unique linear relation between them (Schoenberg [3]), namely

$$(7) \qquad \sum_{\nu=0}^{m-1} a_\nu^{(m)} s(\nu h) = h \sum_{\nu=0}^{m-1} b_\nu^{(m)} s'(\nu h)$$

whose coefficients may be described as follows: writing

$$(8) \qquad Q_m(x) = \frac{1}{(m-1)!} \sum_{i=0}^{m} (-1)^i \binom{m}{i} (x-i)_+^{m-1},$$

which is a B-spline (see [4] and [5]), we have

$$(9) \qquad \begin{aligned} a_\nu^{(m)} &= (m-1)!(Q_m(\nu) - Q_m(\nu+1)), \\ b_\nu^{(m)} &= (m-1)! Q_{m+1}(\nu+1). \end{aligned}$$

From (8) and (9) we easily obtain the numerical values of Table 1.

	$a_\nu^{(m)}$					$b_\nu^{(m)}$				
m \ ν	0	1	2	3	4	0	1	2	3	4
2	−1	1				1/2	1/2			
3	−1	0	1			1/3	4/3	1/3		
4	−1	−3	3	1		1/4	11/4	11/4	1/4	
5	−1	−10	0	10	1	1/5	26/5	66/5	26/5	1/5

TABLE 1.

THEOREM 2. *The values $S_m(\nu h)$ $(\nu = 0, 1, \cdots, n)$ obtained in §1 are precisely the values furnished by the discrete multistep method described by the recurrence relation*

$$(10) \quad \sum_{\nu=0}^{m-1} a_\nu^{(m)} y_{k-m+1+\nu} = h \sum_{\nu=0}^{m-1} b_\nu^{(m)} y'_{k-m+1+\nu} \qquad (k = m - 1, \cdots, n)$$

if we use the starting values

$$(11) \quad y_0 = S_m(0), \qquad y_1 = S_m(h), \cdots, y_{m-2} = S_m((m - 2)h).$$

Notice that our integration method, which appears as a one-step method in §1, according to Theorem 2 furnishes also some of the solutions of the $(m-1)$-step method (10).

3. Quadratic spline functions and the trapezoidal rule. If we consult Table 1 and apply Theorem 2 for $m = 2$, we find the corresponding 1-step method to be

$$y_k - y_{k-1} = (h/2)(y'_{k-1} + y'_k) = (h/2)(f(x_{k-1}, y_{k-1}) + f(x_k, y_k)),$$

which is the trapezoidal rule. Thus by Theorem 2 the trapezoidal rule will give the same discrete solution $S_2(\nu h)$ as the quadratic spline method described in §1 for $m = 2$.

THEOREM 3. *If $f(x, y) \in C^2$ in T, then a constant K can be determined such that for all $h < 2/L$*

$$|S_2(x) - y(x)| < Kh^2, \qquad |S_2'(x) - y'(x)| < Kh^2,$$

$$|S_2''(x) - y''(x)| < Kh,$$

if $x \in [0, b]$, *provided we define the value of the step function* $S_2''(x)$
at a knot x_ν, *by the usual arithmetic mean.*

4. **Cubic spline functions and the Milne-Simpson method.** Again
from Table 1 we find for $m = 3$ the recurrence relation

$$y_k - y_{k-2} = (h/3)(y_{k-2}' + 4y_{k-1}' + y_k')$$
$$= (h/3)(f(x_{k-2}, y_{k-2}) + 4f(x_{k-1}, y_{k-1}) + f(x_k, y_k))$$

which is one way of writing Simpson's rule. Hence, by Theorem 2, the
Milne-Simpson method will furnish the discrete solution $S_3(\nu h)$, pro-
vided that we use as starting values y_0 and $y_1 = S_3(h)$.

THEOREM 4. *If* $f(x, y) \in C^3$ *in* T, *then a constant* K *can be determined
such that for all* $h < 3/L$

$$|S_3(x) - y(x)| < Kh^4, \qquad |S_3'(x) - y'(x)| < Kh^3,$$
$$|S_3''(x) - y''(x)| < Kh^2, \qquad |S_3'''(x) - y'''(x)| < Kh$$

if $x \in [0, b]$, *provided we define the value of the step function* $S_3'''(x)$
at a knot x_ν, *by the usual arithmetic mean.*

An advantage of our procedure of §1 over the use of the recurrence
relation (10) is this: the step size h can be changed at each step, if
necessary, without added complications. This remark adds flexibility
to our version (§1, $m = 3$) of the Milne-Simpson method.

5. **Higher degree spline functions and unstable multistep methods.**
THEOREM 5. *The solutions* $S_m(x)$ *are unstable as* $h \to 0$ *for* $m \geq 4$.

We may apply known criteria for multistep methods (Dahlquist's
theorem; see [1, Theorem 5.5, p. 218]) to the relation (10) in order to
judge the stability of our spline approximation $S_m(x)$. From (9)
and (10) it appears that the stability depends on the zeros of the
"associated polynomial"

$$(12) \qquad \sum_{\nu=0}^{m-1} a_\nu^{(m)} x^\nu = (x-1)(m-1)! \sum_{\nu=0}^{m-2} Q_m(\nu+1)x^\nu.$$

Using (8) and (12) and the fact that the sum of the roots of a poly-
nomial $p(x) = a_0 + \cdots + a_{n-1}x^{n-1} + a_n x^n$ is $-a_{n-1}/a_n$, it is easily seen
that the associated polynomial (12) has at least one root of modulus
greater than unity for $m \geq 4$. Therefore $S_m(x)$ is related by (10) to
an *unstable* multistep method. Numerical results, obtained on the
CDC 1604 at the University of Wisconsin Computing Center, for
the equation $y' = y$, $y(0) = 1$ for $m = 4, 5, 6, 7$, show strong divergence.

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