ON LOCAL TIME FOR MARKOV CHAINS

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The object of this paper is to present one of a class of formulae which express the time spent in an instantaneous state of a Markov chain in terms of the time spent in "neighbouring" states. Though these formulae do lead to new analytic results—some elementary consequences are given in [3]—their main use is in the analysis of sample function behaviour in the neighbourhood of an instantaneous state. Time is *always* shared out "properly" among states in such a neighbourhood. Though the particular theorem stated below refers to a real state b, it has a valid extension to the case when b is an instantaneous *fictitious* state in the sense of Neveu [2]. A detailed account of these topics will appear elsewhere.

The terminology and notation used here are exactly as in Chung [1]; see particularly the appendix for the definition of the functions g and G.

Suppose that $\{x(t): t \geqq 0\}$ is a Borel measurable, well-separable M.C. with minimal state-space I. For any state k in I, define

$$\beta_k(t, \omega) = \mu\{u: 0 \leqq u \leqq t, x(u, \omega) = k\},$$

μ denoting Lebesgue measure.

THEOREM. *Suppose that b is an instantaneous state of $\{x(t)\}$. Suppose also that $\{H_n\}$ is a sequence of subsets of $I - \{b\}$ and that $\{s_n\}$ is a sequence of positive real numbers such that, as $n \to \infty$,*

$$s_n \downarrow 0, \qquad \sum_{j \in H_n} g_{bj}(s_n) \to \infty.$$

Then, for every t,

$$\lim_{n \to \infty} \frac{\sum_{j \in H_n} g_{bj}(s_n)\beta_j(t, \omega)/G_{bj}(\infty)}{\sum_{j \in H_n} g_{bj}(s_n)} = \beta_b(t, \omega)$$

in probability.

Note. Sequences $\{H_n\}$ and $\{s_n\}$ with the stated properties must

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exist. For example, each H_n may be taken to be $I-\{b\}$ and $\{s_n\}$ any sequence tending to zero.

The theorem may be proved by a Tchebycheff argument based on the following calculations.

For $t<\beta_b(\infty, \omega)$, define $\rho(t, \omega)$ by the equation

$$\rho(t, \omega) = \inf\{s: \beta_b(s, \omega) > t\}.$$

Then, for $j\neq b$ and $k\neq b$,

$$\boldsymbol{E}\{\beta_j(\rho(t)) \mid \beta_b(\infty) > t;\, x(0) = b\} = G_{bj}(\infty)F_{jb}(\infty)t$$

and

$$\mathbf{Cov}\{\beta_j(\rho(t)), \beta_k(\rho(t)) \mid \beta_b(\infty) > t;\, x(0) = b\}$$
$$[G_{bj}(\infty)\,{}_bP_{jk}(\infty)F_{kb}(\infty) + G_{bk}(\infty)\,{}_bP_{kj}(\infty)F_{jb}(\infty)]t.$$

The motivation for the theorem was Lévy's formula:

$$\lim_{b\downarrow a} (\mu\{s:\ 0 \leqq s \leqq t,\ a \leqq x(s) \leqq b\})/2(b-a) = T(t, a)$$

for Brownian local time.

References

1. K. L. Chung, *Markov chains with stationary transition probabilities*, Springer-Verlag, Berlin, 1960.

2. J. Neveu, *Sur les états d'entrée et les états fictifs d'un processus de Markov*, Ann. Inst. Henri Poincaré **17** (1962), 323–337.

3. D. Williams, *A note on the Q-matrices of Markov chains*, Z. Wahrscheinlichkeitstheorie **7** (1967), 116–121.

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