SINGULAR INTEGRALS ON HILBERT SPACE

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Introduction. Let H be a real separable Hilbert space and let $1<p<\infty$. Let $y\rightarrow T_y$ denote the strongly continuous representation of the additive group of H as a group of isometries on $L^p(H$, normal distribution) defined by $(T_y f)(x)=f(x-y)D(x,y)$ when f is a bounded tame function on H and

$$D(x, y) = \exp\left[\frac{(x, y)}{p} - \frac{\|y\|^2}{2p}\right].$$

If μ is a Borel measure on H of bounded variation, let μ_p denote the strong integral $\int_H T_y d\mu(y)$. It is the object of this paper to give sufficient conditions on a complex measure μ of bounded variation on H such that if $0<\delta<\rho<\infty$ and if

$$Z^{\delta\rho}(E) = \int_\delta^\rho \mu(E/t)dt/t, \tag{1}$$

then the strong limit of the

$$Z_p^{\delta\rho} = \int_H T_y dZ^{\delta\rho}(y) \tag{2}$$

exists as a bounded operator on $L^p(H)$ as δ tends to zero and ρ tends to infinity.

A theorem of this type extends the Calderon-Zygmund theory of singular integral operators on E_n to infinite dimensions. For if $k(x)\|x\|^{-n}$ is a Calderon-Zygmund kernel and if E is a bounded Borel set which is disjoint from a neighborhood of the origin then $\nu(E) =\int_E k(x)\|x\|^{-n}dx$ satisfies $\nu(tE)=\nu(E)$ for $t>0$; if $g(x)$ is an integrable radial function on E_n with support in a bounded annulus disjoint from a neighborhood of the origin, then $\int_{E_n} g(x)k(x)\|x\|^{-n}dx=0$. When μ satisfies a smoothness condition and $\mu(H)=0$, the set function $\nu(E)=\int_0^\infty \mu(E/t)dt/t$ has these properties.

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The main results.

DEFINITION 1. Let μ be a complex Borel measure of bounded variation on H, let $Z^{\delta\rho}$ be as in Equation (1) and $Z_p^{\delta\rho}$ be as in Equation (2) for $0<\delta<\rho<\infty$. The strong limit, Z_p, of the $Z_p^{\delta\rho}$ as $\delta\to 0$ and $\rho\to\infty$, if it exists, is the singular integral operator determined by μ.

Let n denote the normal distribution on H and let B be a one-one Hilbert-Schmidt operator on H. Then $n \circ B^{-1}$ is a Borel probability measure on H (see [**4**, Corollary 3.2, p. 24]). In what follows we shall consider measures $d\mu(x)=a(x)dn \circ B^{-1}(x)$ where $a(x)$ is absolutely integrable with respect to $n \circ B^{-1}$.

THEOREM 1. *Let* $d\mu(x)=a(x)dn \circ B^{-1}(x)$ *and* $Z^{\delta\rho}$ *be as in Equation* (1) *and consider the operators* $Z_2^{\delta\rho}f=\int_H T_y f dZ^{\delta\rho}(y)$ *on* $L^2(H)$. *If* $a\in \log^+ L(H,\ n \circ B^{-1})$ *and* $\int_H a(x)dn \circ B^{-1}(x)=0$, *then the strong limit*, Z_2, *of the* $Z_2^{\delta\rho}$ *exists as* $\delta\to 0$ *and* $\rho\to\infty$ *and*

$$\|Z_2\| \leqq K_1 \int_H |\,a(x)\,|\ \log^+|\,a(x)\,|\ dn \circ B^{-1}(x) + K_2$$

where K_1 *and* K_2 *are finite constants which do not depend on* $a(x)$. *If in addition*, $a\in L^r(H,\ n \circ B^{-1})$ *for some* $r>1$, *then* $\|Z_2\|\leqq D_r\|a\|_r$ *where* D_r *is a finite constant which depends only on* r *and* $\|a\|_r$ *is the norm of* $a(x)$ *in* $L^r(H,\ n \circ B^{-1})$.

METHOD OF PROOF. Denote by W the Wiener transform (see [**5**, pp. 119–123]) on $L^2(H)$. Then $W(Z_2^{\delta\rho}f)(\cdot)=\hat{Z}^{\delta\rho}(\frac{1}{2}\cdot)^{\sim}W(f)(\cdot)$ where $\hat{Z}^{\delta\rho}(\cdot)^{\sim}$ is the measurable function on H corresponding to the Fourier transform $\hat{Z}^{\delta\rho}(y)$ of the measure $Z^{\delta\rho}$. The $Z_2^{\delta\rho}$ converge strongly on $L^2(H)$ if and only if the $\hat{Z}^{\delta\rho}(\frac{1}{2}\cdot)^{\sim}$ converge boundedly and in measure with respect to the normal distribution on H. The desired conclusions now follow by direct computation.

Let $\mathfrak{F}$ denote the directed set of finite dimensional projections on H and let μ be as in Theorem 1.

DEFINITION 2. For Q in $\mathfrak{F}$, the tame singular integral operator, $(Z \circ Q^{-1})_p$, determined by μ is the strong limit, if it exists, of the tame integral operators $(Z \circ Q^{-1})_p^{\delta\rho}f=\int_H T_y f dZ^{\delta\rho} \circ Q^{-1}(y)$ as δ tends to zero and ρ tends to infinity, where $Z^{\delta\rho}$ is as in Equation (1).

Under the hypotheses of Theorem 1, the tame singular integral operators $(Z \circ Q^{-1})_2$ exists and are uniformly bounded in Q.

THEOREM 2. *Let* $d\mu(x)=a(x)dn \circ B^{-1}(x)$, *suppose* $\mu(H)=0$, *and let* $a\in L^r(H,\ n \circ B^{-1})$ *for some* $r>1$. *Let* Z_2 *be the singular integral operator determined by* μ *as in Theorem* 1 *and let* $\{(Z \circ Q^{-1})_2 \mid Q\in\mathfrak{F}\}$ *be the net of tame singular integral operators determined by* μ. *Then this net converges strongly to* Z_2 *as* Q *tends strongly to the identity through* $\mathfrak{F}$.

For the reflexive L^p-spaces there is

THEOREM 3. *Let $a(x)$, B, μ, $Z^{\delta\rho}$, and $Z_p^{\delta\rho}$ be as above. Then if $a \in L^1(H, n \circ B^{-1})$ is an odd function, the strong limit, Z_p, of the $Z_p^{\delta\rho}$ exists as $\delta \to 0$ and $\rho \to \infty$ and $\|Z_p\| \leqq G_p \|a\|_1$ where G_p is a finite constant which depends only on p. If $a(x)$ is an even tame function in $L^r(H, n \circ B^{-1})$ for some $r > 1$ such that $\int_H a(x) dn \circ B^{-1}(x) = 0$, then the strong limit, Z_p, of the $Z_p^{\delta\rho}$ exists as $\delta \to 0$ and $\rho \to \infty$ and $\|Z_p\| \leqq K(r, p)\|a\|_r$ where $K(r, p)$ is a finite constant which depends on r, p, and the dimension of the base space of $a(x)$.*

REMARK. We have not stated Theorem 3 in the most general form in which we know it to hold since we do not want to introduce new complicated notation in this paper. If μ is an odd Borel measure of bounded variation on H, then μ determines a bounded singular integral operator, Z_p, as above and $\|Z_p\| \leqq G_p \|\mu\|$. In the case in which the function $a(x)$ is even, greater generality is achieved by writing $a(x)$ as a series, $a(x) = \Sigma_i a_i(x)$, where the vector-valued integral of $a_i(x)$ over a certain finite dimensional subspace of H (depending on i) vanishes. The set of functions $a(x)$ for which this series converges absolutely in $L^r(H, n \circ B^{-1})$ forms a Banach space $N^r(H, n \circ B^{-1}) \subset L^r(H, n \circ B^{-1})$ which contains nontame functions and is such that if $a \in N^r(H, n \circ B^{-1})$ then $a(x)$ determines a bounded singular integral operator, Z_p, as above with $\|Z_p\| \leqq KN^r(a)$ where $N^r(a)$ is the norm in $N^r(H, n \circ B^{-1})$ and K is a finite constant which depends only on r and p.

METHOD OF PROOF OF THEOREM 3. When $a(x)$ is an odd function, we apply Minkowski's integral inequality, M. Riesz' theorem on the Hilbert transform, and the dominated convergence theorem.

When $a(x)$ is an even function, a special argument is needed. Let F denote the base of $a(x)$ and G denote the image of F under B. Let P_0 denote the orthogonal projection from H to G. Let $\mathcal{P} = \{P_n\}$ be an ordered sequence $P_0 < P_1 < P_2 < \cdots$ of finite dimensional orthogonal projections which converge strongly to the identity. Consider the tame operators $(Z \circ Q^{-1})_p^{\delta\rho}$ determined by $a(x)$ and $Q \in \mathcal{P}$, $Q > P_0$. We compose $(Z \circ Q^{-1})_p^{\delta\rho}$ with certain Calderon-Zygmund operators $\{R_p^k\}$ on $L^p(H)$ which have odd kernels and which have the properties that $\sum_{k=1}^N R_p^k(R_p^k(Z \circ Q^{-1})_p^{\delta\rho} f) = -(Z \circ Q^{-1})_p^{\delta\rho} f$, where N is the dimension of F, and $R_p^k(Z \circ Q^{-1})_p^{\delta\rho} f = (H \circ Q^{-1})_{pk}^{\delta\rho} f$ where $(H \circ Q^{-1})_{pk}^{\delta\rho} f$ is an approximate tame singular integral operator determined by an odd L^1-function. By the dominated convergence theorem, $(Z \circ Q^{-1})_p^{\delta\rho}$ and $(H \circ Q^{-1})_{pk}^{\delta\rho}$ converge strongly to $Z_p^{\delta\rho}$ and $H_{pk}^{\delta\rho}$, respectively, as Q tends strongly to the identity through $\mathcal{P}$, where $H_{pk}^{\delta\rho}$ is an approximate singular integral operator determined by an odd L^1-function. The result now follows from the case when $a(x)$ is odd and a special estimate for the L^1-norm of the function determining $H^{\delta p}_{pk}$.

Suppose that the Hilbert space, H, is N-dimensional Euclidean space, E_N.

THEOREM 4. *Let $a \in L^r(E_N, n \circ B^{-1})$ for some $r > 1$ and $\int_{E_N} a(x) dn \circ B^{-1}(x) = 0$. Then there is a finite complex constant*

$$A = \int_{E_N} -\log \|x\| a(x) dn \circ B^{-1}(x)$$

and a unique Calderon-Zygmund operator C_p (appropriately transposed to $L^p(E_N, n)$) such that $Af + C_p f = Z_p f$ where Z_p is the singular integral operator of Theorem 3 determined by $a(x)$. C_p has kernel $S(a)(y) \|y\|^{-N}$ where

$$S(a)(\omega) = ((2\pi)^{N/2} \det B \|B\omega^{-1}\|^N)^{-1} \left(\int_0^\infty \Omega(sB\omega^{-1} \|B\omega^{-1}\|^{-1}) \exp \frac{-s^2}{2} s^{N-1} ds \right.$$

for $\|\omega\| = 1$ and $\Omega(x) = a(Bx)$. Furthermore,

$$\|S(a)\|_{L^r(\Sigma, \sigma)} \leqq K(r, N) \|a\|_{L^r(E_N, n \circ B^{-1})}$$

where σ denotes Lebesgue measure on the unit sphere, Σ, and $K(r, N)$ is a finite constant depending only on r and N.

This theorem is proved by direct computation.

REMARK. It follows from Theorem 4 that if the operator B on E_N is the identity operator and if $a(x)$ is homogeneous of degree zero, then the singular integral operator Z_p determined by $a(x)$ is the Calderon-Zygmund operator C_p and C_p has kernel const. $a(y)\|y\|^{-N}$. Furthermore, every Calderon-Zygmund operator arises in this way.

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