DE RHAM THEOREMS ON SEMIANALYTIC SETS

BY M. E. HERRERA[1]

Semianalytic sets are subsets of real analytic manifolds locally defined by inequalities of real analytic functions. We refer to [5] for precise definitions and properties about them. Let M be a semi-analytic set in the real analytic manifold X.

We consider in this note the complexes of differential forms and currents induced on M by the smooth forms and currents of X, and relate them to the real cohomology and homology of M. The following version of de Rham's theorems holds: there is an epimorphism from the cohomology of the forms on M onto the cohomology of M, and there is a monomorphism from the homology of M into the homology of the currents on M. In general these maps are not isomorphisms, even on algebraic sets in R^2. These results answer a question posed by Norguet [7].

We show in the first section that homology and cohomology classes of M can be represented by semianalytic chains and cochains. This is used in the second section to prove de Rham's theorems. In the third section an example is given in which the above maps are not isomorphisms, together with some particular remarks on the Poincaré lemma.

It is supposed throughout this note that X is paracompact and that M is closed in X and has dimension p. Then the set M^* of the regular points of M is an analytic submanifold of X and the singular set $\partial M = M - M^*$ is semianalytic in X with dimension dim $\partial M < p$. If N is semianalytic and locally closed in X with dim $N \leq q$, then $bN = \overline{N} - N$ is semianalytic and closed in X with dim $bN < q$.

Unless stated otherwise, K is a principal ideal domain. If ϕ is a family of supports on the locally compact space Y and $\mathfrak{F}$ is a sheaf of K-modules on Y, then $H_*(Y; \mathfrak{F})(H_*^\phi(Y; \mathfrak{F}))$ denotes the Borel-Moore homology of Y with coefficients in $\mathfrak{F}$ and closed supports (supports in ϕ) [1]. If $F \subset Y$ is closed, there is an exact sequence

$$\cdots \to H_q(F; K) \xrightarrow{i_{F,Y}} H_q(Y, K) \xrightarrow{j^{Y,Y-F}} H_q(Y - F; K)$$

$$\xrightarrow{\partial_{Y,F}} H_{q-1}(F; K) \to \cdots .$$

Details and complements will be published later. I am indebted to L. Bungart for a useful correspondence on the subject.

1. **Semianalytic chains and cochains.** Denote by $\mathcal{P}\mathfrak{J}_q(M; K)$ the set of pairs (N, c) such that $N \subset M$ is a locally closed semianalytic set in X with dim $N \leq q$ and $c \in H_q(N; K)$. The elements of $\mathcal{P}\mathfrak{J}_q(M; K)$ are called semianalytic prechains of M. We define a commutative and associative operation on $\mathcal{P}\mathfrak{J}_q(M; K)$ as follows: if (N_1, c_1) and (N_2, c_2) are prechains, then

$$(N_1, c_1) + (N_2, c_2) = (N, c^1 + c^2),$$

where $N = N_1 \cup N_2 - L$, $L = bN_1 \cup bN_2$ and c^s is the image of c_s by the maps

$$H_q(N_s; K) \xrightarrow{j^{N_s, N_s - L}} H_q(N_s - L; K) \xrightarrow{i_{N_s - L, N}} H_q(N; K) \qquad (s = 1, 2).$$

A product by elements in K, linear with respect to this addition, is also defined by $k(N, c) = (N, kc)$, for each $k \in K$ and each prechain (N, c).

Consider the equivalence relation on $\mathcal{P}\mathfrak{J}_q(M; K)$

$$(N_1, c_1) \sim (N_2, c_2) \Leftrightarrow (N_1, c_1) + (N_2, - c_2) = (N, 0),$$

where 0 is the zero element of $H_q(N; K)$. The addition and product on $\mathcal{P}\mathfrak{J}_q(M; K)$ induce a K-module structure on the quotient $\mathfrak{J}_q(M; K) = \mathcal{P}\mathfrak{J}_q(M; K)/\sim$, whose elements are called q-semianalytic chains of M. The chain represented by a prechain (N, c) is denoted by $[N, c]$. $\mathfrak{J}_q(M; K) = 0$ if $q < 0$ or $q > \dim M$.

A boundary homomorphism $\partial \colon \mathfrak{J}_q(M; K) \to \mathfrak{J}_{q-1}(M; K)$ such that $\partial \circ \partial = 0$ is induced on $\mathfrak{J}_*(M) = \sum(\mathfrak{J}_q(M; K) : q \in \mathbf{Z})$ by the maps

$$\mathcal{P}\mathfrak{J}_q(M; K) \to \mathcal{P}\mathfrak{J}_{q-1}(M; K), \quad (N, c) \to (bN, \partial_{N, bN}(c)), \quad (q \in \mathbf{Z}).$$

Let now M' be an open subset of M and U any open set in X such that $U \cap M = M'$. The module $\mathfrak{J}_*(M'; K)$ is defined, if we consider M' as a closed semianalytic set in U, and is independent of the particularly chosen set U. Moreover, if $M'' \subset M'$ is also open in M, a restriction homomorphism $\mathfrak{J}_*(M'; K) \to \mathfrak{J}_*(M''; K)$, compatible with boundaries, is induced by the map

$$\mathcal{P}\mathfrak{J}_q(M'; K) \to \mathcal{P}\mathfrak{J}_q(M'', K), (N, c) \to (N \cap M'', j^{N, N \cap M''}(c)).$$

It is clear that $\mathfrak{J}_* \colon M' \to \mathfrak{J}_*(M'; K)$ (M' open in M) is a differential presheaf.

1.1. THEOREM. *The presheaf $\mathfrak{J}_*$ of semianalytic chains of M with*

coefficients in K is a soft differential sheaf. For any sheaf of K-modules $\mathfrak{F}$ and any family of supports ϕ on M, there exists a natural isomorphism

$$H_q^\phi(M; \mathfrak{F}) \simeq H_q(\Gamma_\phi(\mathfrak{I}_* \otimes \mathfrak{F})).$$

PROOF. A canonical monomorphism $H_q(\Gamma_\phi(\mathfrak{I}_*)) \to H_q^\phi(M; K)$ is easy to construct. To see that it is onto, Lojasiewicz's triangulation theorem [5] can be applied. To prove the general coefficient case, consider the covariant functors $S_q: M' \to \Gamma_c(\mathfrak{I}_q | M')$ (M' open in M) together with the deduced boundaries $\partial_q: S_q \to S_{q-1}$ and denote $B_q = \operatorname{Im} \partial_{q+1}$, $Z_q = \ker \partial_q$ and $H_q(S_*) = Z_q/B_q$ ($q \in \mathbf{Z}$). A natural homomorphism $\eta: H_0(S_*) \to K$ can be defined and, if T is any one of the functors $\ker \eta$, $K/\operatorname{Im} \eta$ and $H_q(S_*)$ ($q > 0$), the following property can be proved to hold by means of the topological local contractibility of M: for each open set $M' \subset M$ and $y \in M'$, there exists an open set M'' such that $y \in M'' \subset M'$ and the homomorphism $T(M'') \to T(M')$ is trivial. Then the theorem follows from a general result by G. Bredon [2, V (11.15)].

Let now $L \subset M$ be another closed semianalytic set in X, ϕ a family of supports on M and $\phi' = \{A \cap (M-L) : A \in \phi\}$ its restriction to $M - L$. Define the group $\Gamma_\phi \mathfrak{I}_*(M, L; K)$ of relative semianalytic chains of M module L, with supports in ϕ, by the exact sequence

$$0 \to \Gamma_\phi \mathfrak{I}_*(L; K) \to \Gamma_\phi \mathfrak{I}_*(M; K) \to \Gamma_\phi \mathfrak{I}_*(M, L; K) \to 0.$$

1.2. COROLLARY. *There exists a natural isomorphism*

$$H_q(\Gamma_\phi \mathfrak{I}_*(M, L; K)) \simeq H_q^{\phi'}(M - L; K) \qquad (q \in \mathbf{Z}).$$

We suppose in the following that $K = \mathbf{R}$, the field of real numbers, and omit it when no confusion can arise. Consider the presheaves $\mathfrak{I}^q: M' \to \operatorname{Hom}(S_q(M'), \mathbf{R})$ (M' open in M) on M with differentials $\mathfrak{I}^q \to \mathfrak{I}^{q+1}$ ($q \in \mathbf{Z}$) deduced from the boundaries of S_*. The elements in $\mathfrak{I}^q(M)$ are called semianalytic cochains of M, and the previous theorem implies

1.3. PROPOSITION. $\mathfrak{I}^* = \sum_{q \in \mathbf{Z}} \mathfrak{I}^q$ *is a resolution of $\mathbf{R}$ on M by flabby sheaves.*

2. **de Rham's theorems.** Let $\mathcal{E}_X^*(\mathcal{E}_{M^*}^*)$ be the sheaf of germs of smooth differentials forms on X (on M^*) and $\mathcal{E}_{X,M}^*$ the subsheaf of $\mathcal{E}_X^*$ of the germs whose image by the map $\mathcal{E}_X^* \to \mathcal{E}_{M^*}^*$ induced by the embedding $M^* \to X$ is zero. Define the sheaf $\mathcal{E}^* = \mathcal{E}_M^*$ of germs of differential forms on M as the restriction to M of the quotient sheaf $\mathcal{E}_X^*/\mathcal{E}_{X,M}^*$. $\mathcal{E}^*$ is a soft differential sheaf, with differential d deduced from the exterior differential of $\mathcal{E}_X^*$, and there is an exact sequence

$$0 \rightarrow \Gamma_c \overset{*}{\mathcal{E}}_{X,M} \rightarrow \Gamma_c \overset{*}{\mathcal{E}}_X \rightarrow \Gamma_c \mathcal{E}^* \rightarrow 0.$$

Consider the usual locally convex inductive topology on the space $\Gamma_c \mathcal{E}_X^* = \mathfrak{D}^*(X)$ of the forms on X with compact support, and the quotient topology on $\Gamma_c \mathcal{E}^* = \mathfrak{D}^*(M)$. The space $\mathfrak{D}_q'(M)$ of the q-currents on M is defined as the topological dual of $\mathfrak{D}^q(M)$, and a border homomorphism $b: \mathfrak{D}_q'(M) \rightarrow \mathfrak{D}_{q-1}'(M)$ such that $b^2 = 0$ is given on $\mathfrak{D}_*'(M)$ by $bT(\alpha) = T(d\alpha)$, for each $T \in \mathfrak{D}_q'(M)$ and $\alpha \in \mathfrak{D}^{q-1}(M)$ $(q \in \mathbf{Z})$. The sheaf $\mathfrak{D}_*'$ of germs of currents on M, constructed obviously, is a soft differential sheaf. $\mathfrak{D}^q$ and $\mathfrak{D}_q'$ are zero if $q < 0$ or $q > \dim M$.

Let now M' be open in M, U open in X such that $U \cap M = M'$ and $[N, c] \in \mathfrak{I}_q(M')$. An integration current $I(N, c) \in \mathfrak{D}_q'(U)$ has been associated in [4, II, A, 2.1] to the prechain (N, c). This current depends only on the class $[N, c]$, and defines a current $I[N, c] \in \mathfrak{D}_q'(M')$. The so defined maps $\mathfrak{I}_q(M') \rightarrow \mathfrak{D}_q'(M')$ $(M'$ open in M; $q \in Z)$ give a sheaf homomorphism $I: \mathfrak{I}_* \rightarrow \mathfrak{D}_*'$, whose compatibility with boundaries is assured by [4, II, B, 2.1]. If $[N, c] \in S_q(M')$, $I[N, c]$ has compact support, so that the semianalytic cochain $\lambda_{M'}(\alpha): [N, c] \rightarrow I[N, c](\alpha)$ exists for each $\alpha \in \mathcal{E}^q(M')$. We deduce a differential sheaf homomorphism $\lambda: \mathcal{E}^* \rightarrow \mathfrak{I}^*$.

For any family of supports ϕ in M, consider the homomorphisms

$$\lambda_2: E_2^{pq} = H_\phi^p(M; \mathfrak{K}^q(\mathcal{E}^*)) \rightarrow \tilde{E}_2^{pq} = H_\phi^p(M; \mathfrak{K}^q(\mathfrak{I}^*))$$

induced by λ between the spectral sequences of $\mathcal{E}^*$ and $\mathfrak{I}^*$ ([3, II, 4.6.1]). By 1.3, $\mathfrak{K}^q(\mathfrak{I}^*) = 0$ if $q \neq 0$, and λ defines an isomorphism between $\mathfrak{K}^0(\mathcal{E}^*) \simeq R$ and $\mathfrak{K}^0(\mathfrak{I}^*) \simeq R$. We conclude that λ_2 is an epimorphism, as is $\lambda_\infty: E_\infty \rightarrow \tilde{E}_\infty$, whence the first part of the following

2.1. THEOREM. *For each family of supports ϕ on M, there is a canonical epimorphism*

$$\bar{\lambda}: H^q(\Gamma_\phi \mathcal{E}^*(M)) \rightarrow H^q(\Gamma_\phi \mathfrak{I}^*(M)) \simeq H_\phi^q(M; \mathbf{R})$$

and a canonical monomorphism

$$\bar{I}: H_q^\phi(M; \mathbf{R}) \simeq H_q(\Gamma_\phi \mathfrak{I}_*(M)) \rightarrow H_q(\Gamma_\phi \mathfrak{D}_*'(M)) \qquad (q \in \mathbf{Z}).$$

The second part follows from the first. In particular we deduce a monomorphism $\mathfrak{K}_*(M; \mathbf{R}) \rightarrow \mathfrak{K}(\mathfrak{D}_*')$, where the homology sheaf $\mathfrak{K}_*(M; \mathbf{R})$ of M may not be locally trivial or concentrated in the dimension of M.

3. **Poincaré lemma.** If the Poincaré lemma holds for $\mathcal{E}_M^*$, that is, if $\mathfrak{K}^0(\mathcal{E}_M^*) \simeq \mathbf{R}$ (which is always the case) and $\mathfrak{K}^q(\mathcal{E}_M^*) = 0$ for $q > 0$,

then $\bar{\lambda}$ is an isomorphism, as $\bar{I}$ can be proved to be. This is not in general the case. Consider for example the algebraic set $M_0 = (H(x, y) = 0) \subset R^2$, where $H(x, y) = y(y - x^3)(y - x^4)$, and the germ $\alpha \in \mathcal{E}^1_{M_0}$ defined by the form ydx at $0 \in M_0$. Suppose that a germ $\bar{f} \in \mathcal{E}^0_{M_0}$ exists such that $df^- = \alpha$. This means that a smooth function f, defined in some neighborhood $U \subset R^2$ of 0, satisfies $df - ydx \mid M_0^* \cap U = 0$.

Malgrange's differentiable preparation theorem [6] implies that $f = QH + R$ in some neighborhood U' of 0, where Q and R are smooth and $R(x, y) = b_1(x)y^2 + b_2(x)y + b_3(x)$, and we must necessarily have $dR - ydx \mid M_0^* \cap U' = 0$, since $d(QH) \mid M_0^* = 0$. By means of the coordinate maps defined on the branches of M_0^* by the projection $(x, y) \to x$, this last equation reduces to the following system of differential equations:

$$x^6 b_1' + 6x^5 b_1 + x^3 b_2' + 3x^2 b_2 = x^3$$

$$x^8 b_1' + 8x^7 b_1 + x^4 b_2' + 4x^3 b_2 = x^4$$

which is easy to check has no smooth solutions at $x = 0$. We conclude that $\mathcal{H}^1(\mathcal{E}^*_{M_0}) \neq 0$ at $0 \in M_0$ and can deduce that $\bar{\lambda}$ is not an isomorphism.

It can be proved that, if $\dim M = 1$, then the equation of germs $df = \alpha$ $(\alpha \in \mathcal{E}^1_M)$ has always a solution of the form $f = g \cdot D^{-1}$, where $g \in \mathcal{E}^0_M$ and D is a real analytic function of one variable with a zero at ∂M. A similar property holds for complex analytic sets of dimension 1. As a consequence, $\mathcal{H}^1(\mathcal{E}^*_M)$ has finite dimensional stalks and $H^1(\Gamma_c \mathcal{E}^*_M)$ has finite dimension locally. This implies that $\bar{I}$ is an isomorphism if and only if $\bar{\lambda}$ is, so that in the above example $\bar{I}$ is not surjective.

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