A COMMUTATIVE SEMISIMPLE ANNIHILATOR BANACH ALGEBRA WHICH IS NOT DUAL

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Communicated by E. Hewitt, December 7, 1966

Barnes [1] has constructed an example of a commutative semisimple normed annihilator algebra which is not a dual algebra. His example is not complete and when completed acquires a nonzero radical. In this paper we construct an example which is complete. The theory of annihilator algebras is developed e.g. in [2].

We put $\alpha_i = (1+(1+i)^{1/2})^{-2}$ for $i \geqq 1$ and denote by A_0 the algebra of doubly infinite sequences a with $a_i = 0$ for all but a finite number of values of i, with coordinatewise addition and multiplication. We define a norm on A_0 by

$$\|a\| = 3\left(\sum_{n\leqq 0} |a_n|^2\right)^{1/2} + 3 \sup_{n>0} \left| a_n\alpha_n^{-1} - \sum_{j=-n}^{0} a_j \right|.$$

This is easily seen to be a linear space norm on A_0 and we have that

(i)
$$\left(\sum_{n\leqq 0} |a_n b_n|^2\right)^{1/2} \leqq \left(\sum_{n\leqq 0} |a_n|^2\right)^{1/2}\left(\sum_{n\leqq 0} |b_n|^2\right)^{1/2} \leqq \tfrac{1}{9}\|a\|\,\|b\|;$$

(ii) if $n > 0$,

$$\tfrac{1}{3}\|a\| \geqq \left| a_n\alpha_n^{-1} - \sum_{j=-n}^{0} a_j \right| \geqq |a_n|\,\alpha_n^{-1} - (n+1)^{1/2}\tfrac{1}{3}\|a\|$$

so that

$$|a_n|\,\alpha_n^{-1} \leqq \tfrac{1}{3}(1+(n+1)^{1/2})\|a\|$$

and

$$|a_n b_n \alpha_n^{-1}| \leqq \tfrac{1}{9}\alpha_n(1+(n+1)^{1/2})^2\|a\|\,\|b\| = \tfrac{1}{9}\|a\|\,\|b\|;$$

(iii)
$$\left|\sum_{j=-n}^{0} a_j b_j\right| \leqq \left(\sum_{j=-n}^{0} |a_j|^2\right)^{1/2}\left(\sum_{j=-n}^{0} |b_j|^2\right)^{1/2} \leqq \tfrac{1}{9}\|a\|\,\|b\|.$$

The submultiplicative property of $\|\cdot\|$ follows easily from (i), (ii) and (iii).

Consider now the space $l_2(-\infty, 0)\oplus c(1, \infty)$, which we consider as a space of doubly infinite sequences, with norm as the sum of the l_2 norm and the sup norm. For $a\in A_0$ define $T_0\, a$ by

$$(T_0a)_n = 3^{1/2}a_n \qquad \text{for } n \leqq 0,$$
$$= 3a_n\alpha_n^{-1} - 3\sum_{j=-n}^{0} a_j \qquad \text{for } n > 0.$$

T_0 is then a linear isometry; $A_0\to l_2(-\infty, 0)\oplus c(1, \infty)$. The multiplicative linear functionals on A_0 are

$$\phi_i(a) = a_i$$

and if the functionals ψ_i on $l_2\oplus c$ are defined by

$$\psi_i(c) = c_i/3^{1/2} \qquad \text{for } i \leqq 0,$$
$$= \left(c_i + (3^{1/2})\sum_{j=-i}^{0} c_j\right)\alpha_i/3 \qquad \text{for } i > 0,$$

then the ψ_i are continuous and $T_0^*\psi_i=\phi_i$. The set $\{\psi_i;\ i\in \boldsymbol{Z}\}$ is clearly total on $l_2\oplus c$.

Let now A be the completion of A_0; T_0 extends to an isometry T; $A\to l_2\oplus c$, the ϕ_i extend to multiplicative linear functionals on A and $\phi_i=T^*\psi_i$. Since the ψ_i are total on $l_2\oplus c$, the ϕ_i are total on A and A is a semisimple Banach algebra. Writing $a_i=\phi_i(a)$ for $a\in A$ we can consider the elements of A as doubly infinite sequences and the two ways in which an element of A_0 becomes a sequence give the same sequence.

If δ_i is the sequence in A_0 with $(\delta_i)_j=\delta_{ij}$ (the Kronecker symbol) then $a\delta_i=a_i\delta_i$ for all a in A so that if J is an ideal in A either $\delta_i\in J$ or $\delta_iJ=\{0\}$. Thus if J is a closed ideal in A with zero annihilator then all the δ_i are in J, $A_0\subset J$ and $J=A$. Hence A is an annihilator algebra.

The span J_0 of the set $\{\delta_i;\ i>0\}$ is an ideal in A and thus so is its closure J. The annihilator of J is

$$K = \{b\colon b \in A,\ b_i = 0 \text{ for } i > 0\}$$

and the annihilator of K is

$$\tilde{J} = \{c\colon c \in A,\ c_i = 0 \text{ for } i \leqq 0\}.$$

The norm on $\tilde{J}$ is given by $\|c\|=\sup|c_i/\alpha_i|$ and since $a_n=o(\alpha_n)$ as $n\to\infty$ for $a\in J_0$ we have $a_n=o(\alpha_n)$ for $a\in J$. Define a sequence x_n from A_0 by

$$
\begin{aligned}
(x_n)_i &= -1/(n+1), \qquad -n \leqq i \leqq 0,\\
&= 0, \qquad i < -n,\\
&= \left(1 + \sum_{j=-i}^{0} x_{nj}\right)\alpha_i, \qquad i > 0.
\end{aligned}
$$

Then, since the supremum term is 0,

$$
\begin{aligned}
\|x_m - x_n\| &= 3\left(\sum_{i\leq 0} |x_{mi} - x_{ni}|^2\right)^{1/2}\\
&\leqq 3\left(\sum_{i\leq 0} |x_{mi}|^2\right)^{1/2} + 3\left(\sum_{i\leqq 0} |x_{ni}|^2\right)^{1/2}\\
&= 3(m+1)^{-1/2} + 3(n+1)^{-1/2},
\end{aligned}
$$

so that x_n converges to a limit y in A. We have

$$
\begin{aligned}
y_i = \lim_n x_{ni} &= 0, \qquad \text{if } i \leqq 0,\\
&= \alpha_i, \qquad \text{if } i > 0.
\end{aligned}
$$

Clearly $y \in \tilde{J}$ but $y_i = \alpha_i \neq o\ (\alpha_i)$ so that $y_i \notin J$, the ideal J is not an annihilator and A is not a dual algebra.

The question of the existence of simple annihilator Banach algebras which are not dual remains open!

References

1. B. A. Barnes, *An annihilator algebra which is not dual*, Bull, Amer. Math. Soc. **71** (1965), 573–576.

2. C. E. Rickart, *General theory of Banach algebras*, Van Nostrand, New York, 1960.

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