DIFFERENCES OF MEANS

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1. Let $q_1, q_2, \cdots, q_n$ be positive numbers with $\sum_{k=1}^n q_k=1$. For every sequence $(x_1, x_2, \cdots, x_n)$ with all $x_k>0$ and for every real r, consider the *mean of order* r, $M_r(x_1, x_2, \cdots, x_n)$, defined as $(\sum_{k=1}^n q_k x_k^r)^{1/r}$ if $r\neq 0$, and as $\prod_{k=1}^n x_k^{q_k}$ if $r=0$. For given positive $x_1, x_2, \cdots, x_n$, it is known (see, e.g. [**3**, p. 17], or [**11**, p. 26]) that $M_r(x_1, x_2, \cdots, x_n)$ is strictly increasing with r (except when $x_1 = x_2 \cdots = x_n$), and consequently if $r<s$, then

$$1 \leqq M_s(x_1, x_2, \cdots, x_n)/M_r(x_1, x_2, \cdots, x_n), \tag{1}$$

$$0 \leqq M_s(x_1, x_2, \cdots, x_n) - M_r(x_1, x_2, \cdots, x_n). \tag{2}$$

2. A natural question to ask is, whether one can give *upper bounds* for the right hand sides of (1) and (2) under, say, the hypothesis that $A \leqq x_j \leqq B, j=1, 2, \cdots, n$, where A and B are constants $(0<A<B)$. Such an upper bound for the ratio in (1) was given by Cargo and Shisha in [**4**], a paper which served as a motivation and starting point of a considerable amount of further work by various authors.

3. The main purpose of the present note is to give an upper bound for the difference in (2) under the restriction on the x_j stated in §2. As applications, we shall obtain a number of inequalities, including "complements" to the classical inequalities of Cauchy and Hölder. Full proofs omitted here are to be found in [**15**].

4. In this section, $q_1, q_2, \cdots, q_n$ are fixed (though arbitrary) positive numbers with $\sum_{k=1}^n q_k=1$, and for every sequence $(x_1, x_2, \cdots, x_n)$ with all $x_k>0$ and every real r, $M_r(x_1, x_2, \cdots, x_n)$ is as in §1.

THEOREM 1. *Let r, s, A, B be given reals $(0<A<B$, $r<s)$, and let I denote the n-dimensional* cube $\{(x_1, x_2, \cdots, x_n): A\leqq x_k \leqq B, k=1, 2, \cdots, n\}$. *Then throughout I,*

$$M_s(x_1, x_2, \cdots, x_n) - M_r(x_1, x_2, \cdots, x_n) \leqq \Delta, \tag{3}$$

where Δ *is*

$$[\theta B^s + (1-\theta)A^s]^{1/s} - [\theta B^r + (1-\theta)A^r]^{1/r} \qquad \text{if } rs \neq 0, \tag{4}$$
$$[\theta B^s + (1-\theta)A^s]^{1/s} - B^\theta A^{1-\theta} \qquad \text{if } r = 0,$$

and

$$B^\theta A^{1-\theta} - [\theta B^r + (1-\theta)A^r]^{1/r} \qquad \textit{if } s = 0.$$

θ is defined as follows. Let

$$h(x) \equiv x^{1/s} - (ax+b)^{1/r} \text{ with } a = \frac{B^r - A^r}{B^s - A^s},$$

$$b = \frac{B^sA^r - B^rA^s}{B^s - A^s}, \qquad \textit{if } rs \neq 0,$$

$$h(x) \equiv x^{1/s} - A(B/A)^{(x-A^s)/(B^s-A^s)} \qquad \textit{if } r = 0,$$

and

$$h(x) \equiv -x^{1/r} + A(B/A)^{(x-A^r)/(B^r-A^r)} \qquad \textit{if } s = 0.$$

Let J denote the open interval joining A^s to B^s if $s \neq 0$, and let $J = (B^r, A^r)$ if $s = 0$. There is an $x^* \in J$ such that $h(x) < h(x^*)$ for every x ($\in J$, $\neq x^*$). (Observe that if $rs \neq 0$, then $ax+b > 0$ at the end points of J and, therefore, throughout J.) We set

$$\theta = (x^* - A^s)/(B^s - A^s) \qquad \textit{if } s \neq 0,$$

and

$$\theta = (x^* - A^r)/(B^r - A^r) \qquad \textit{if } s = 0.$$

Equality in (3) for a point $(x_1, x_2, \cdots, x_n) \in I$ holds if and only if there exists a subsequence $(k_1, k_2, \cdots, k_p)$ of $(1, 2, \cdots, n)$ such that $\sum_{m=1}^{p} q_{k_m} = \theta$, $x_{k_m} = B$ $(m = 1, 2, \cdots, p)$, and $x_k = A$ for every k distinct from all k_m. Finally, if $s \geqq 1$, then x^* is the unique solution of $h'(x) = 0$ in J.

5. Here is an outline of a proof of Theorem 1. Suppose $rs \neq 0$. Then [12] throughout (A, B),

$$r(x^r - ax^s - b) > 0.$$

Let $(x_1, x_2, \cdots, x_n)$ be a point of I. Then

$$r\left[\left(\sum_{k=1}^{n} q_k x_k^r\right) - a\left(\sum_{k=1}^{n} q_k x_k^s\right) - b\right] \geqq 0,$$

and so,

$$\begin{aligned} M_s(x_1, x_2, \cdots, x_n) &- M_r(x_1, x_2, \cdots, x_n) \\ &\leqq \left(\sum_{k=1}^{n} q_k x_k^s\right)^{1/s} - \left[a\left(\sum_{k=1}^{n} q_k x_k^s\right) + b\right]^{1/r} \\ &= h\left(\sum_{k=1}^{n} q_k x_k^s\right). \end{aligned}$$

One shows that there is a unique point in the closure of J where $h(x)$ attains its maximum there, and this point, x^*, belongs to J. Thus,

$$\begin{aligned} M_s(x_1, x_2, \cdots, x_n) - M_r(x_1, x_2, \cdots, x_n) &\leqq h(x^*) \\ &= h(\theta B^s + (1-\theta)A^s) \\ &= [\theta B^s + (1-\theta)A^s]^{1/s} - [\theta B^r + (1-\theta)A^r]^{1/r} = \Delta. \end{aligned}$$

In case $s \geqq 1$, one can also prove the theorem by the method used in [4] to obtain an upper bound for the right hand side of (1). Namely, suppose $n > 1$, $r \neq 0$, and set $F(x_1, x_2, \cdots, x_n) \equiv M_s(x_1, x_2, \cdots, x_n) - M_r(x_1, x_2, \cdots, x_n)$; one shows that a point of I where F attains its maximum in I must be a vertex of I. Thus, the last difference is bounded in I by $\max\{[xB^s+(1-x)A^s]^{1/s} - [xB^r+(1-x)A^r]^{1/r}: 0 \leqq x \leqq 1\}$ which equals Δ.

REMARK. If $r \neq 0$, $s \geqq 1$, then θ of Theorem 1 is the unique solution in (0, 1) of

$$\frac{\gamma^s - 1}{s}[x(\gamma^s - 1) + 1]^{(1/s)-1} - \frac{\gamma^r - 1}{r}[x(\gamma^r - 1) + 1]^{(1/r)-1} = 0$$
$$(\gamma = B/A).$$

6. Here are two examples.

EXAMPLE 1. Let $q_1, q_2, \cdots, q_n$ be positive numbers with $\sum_{k=1}^n q_k = 1$, let $0 < A < B$, and set $\gamma = B/A$. Let $A \leqq x_k \leqq B$, $k = 1, 2, \cdots, n$. By Theorem 1 and by the preceding Remark,

$$\left(\sum_{k=1}^n q_k x_k^2\right)^{1/2} - \sum_{k=1}^n q_k x_k \leqq [\theta B^2 + (1-\theta)A^2]^{1/2} - [\theta B + (1-\theta)A],$$

where θ is the unique solution in (0, 1) of

$$\tfrac{1}{2}(\gamma^2 - 1)[x(\gamma^2 - 1) + 1]^{-1/2} - (\gamma - 1) = 0.$$

A short calculation yields

$$\left(\sum_{k=1}^n q_k x_k^2\right)^{1/2} - \sum_{k=1}^n q_k x_k \leqq \frac{(B-A)^2}{4(B+A)}. \tag{5}$$

Equality holds in (5) if and only if there exists a subsequence $(k_1, k_2, \cdots, k_p)$ of $(1, 2, \cdots, n)$ such that

$$\sum_{m=1}^p q_{k_m} = \frac{B + 3A}{4(B+A)} (=\theta), \qquad x_{k_m} = B(m = 1, 2, \cdots, p),$$

and $x_k = A$ for every k distinct from all k_m.

EXAMPLE 2. Let $q_1, q_2, \cdots, q_n, A, B, \gamma, x_1, x_2, \cdots, x_n$ be as in Example 1. By Theorem 1 and by the last Remark,

$$\sum_{k=1}^{n} q_k x_k - \left(\sum_{k=1}^{n} q_k/x_k\right)^{-1} \leqq \theta B + (1-\theta)A - [\theta B^{-1} + (1-\theta)A^{-1}]^{-1},$$

where θ is the unique solution in (0, 1) of

$$\gamma - 1 + (\gamma^{-1} - 1)[x(\gamma^{-1} - 1) + 1]^{-2} = 0.$$

Solving for θ and substituting in the last inequality, one gets

$$\left(\sum_{k=1}^{n} q_k x_k\right) - \left(\sum_{k=1}^{n} q_k/x_k\right)^{-1} \leqq (B^{1/2} - A^{1/2})^2. \tag{6}$$

Equality holds in (6) if and only if there exists a subsequence $(k_1, k_2, \cdots, k_p)$ of $(1, 2, \cdots, n)$ such that $\sum_{m=1}^{p} q_{k_m} = (1+\gamma^{-1/2})^{-1}$, $x_{k_m} = B$ $(m = 1, 2, \cdots, p)$, and $x_k = A$ for every k distinct from all k_m.

(6) can be obtained directly. For $k = 1, 2, \cdots, n$, we have,

$$x_k - (A + B) + ABx_k^{-1} = (x_k - A)(x_k - B)x_k^{-1} \leqq 0,$$

$$q_k x_k \leqq (A + B)q_k - ABq_k x_k^{-1}.$$

So,

$$\begin{aligned}
&\left(\sum_{k=1}^{n} q_k x_k\right) - \left(\sum_{k=1}^{n} q_k x_k^{-1}\right)^{-1} \\
&\leqq A + B - AB\left(\sum_{k=1}^{n} q_k x_k^{-1}\right) - \left(\sum_{k=1}^{n} q_k x_k^{-1}\right)^{-1} \\
&= A + B - \left[\left\{AB\sum_{k=1}^{n} q_k x_k^{-1}\right\}^{1/2} - \left\{\sum_{k=1}^{n} q_k x_k^{-1}\right\}^{-1/2}\right]^2 - 2(AB)^{1/2} \\
&\leqq A + B - 2(AB)^{1/2} \\
&= (B^{1/2} - A^{1/2})^2.
\end{aligned}$$

One can also derive from this proof the necessary and sufficient condition given above for equality in (6). (Compare the method of this proof with Diaz and Metcalf [9, §2, Remark 3].)

7. Let $0 < m_1 \leqq a_j \leqq M_1$, $0 < m_2 \leqq b_j \leqq M_2$, $j = 1, 2, \cdots, n$, $n \geqq 1$, $m_1 m_2 < M_1 M_2$, and let $\xi_1, \xi_2, \cdots, \xi_n$ be real numbers $\neq 0$. Set $q_j = a_j b_j \xi_j^2 / \sum_{k=1}^{n} a_k b_k \xi_k^2$, $x_j = a_j/b_j$ $(j = 1, 2, \cdots, n)$. Observe that

$$0 < m_1/M_2 \leqq x_j \leqq M_1/m_2 \qquad (j = 1, 2, \cdots, n).$$

By (6),

$$(7)\qquad \left[\left(\sum_{j=1}^{n} a_j^2\xi_j^2\right)\Big/ \sum_{j=1}^{n} a_jb_j\xi_j^2\right]-\left[\left(\sum_{j=1}^{n} a_jb_j\xi_j^2\right)\Big/ \sum_{j=1}^{n} b_j^2\xi_j^2\right] \leqq [(M_1/m_2)^{1/2}-(m_1/M_2)^{1/2}]^2.$$

One obtains also a necessary and sufficient condition for equality in (7). Since the left hand side of (7) is $\geqq 0$ by Cauchy's inequality, (7) may be considered a "complementary" inequality to Cauchy's. Taking $\xi_1=\xi_2 \cdots \xi_n=1$, we obtain

$$\left[\left(\sum_{j=1}^{n} a_j^2\right)\Big/ \sum_{j=1}^{n} a_jb_j\right]-\left[\left(\sum_{j=1}^{n} a_jb_j\right)\Big/ \sum_{j=1}^{n} b_j^2\right] \leqq [(M_1/m_2)^{1/2}-(m_1/M_2)^{1/2}]^2.$$

8. We give now a complement to Hölder's inequality.

THEOREM 2. *Let $p>1$, $p^{-1}+q^{-1}=1$, $0<A<B$, and let $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n$ be positive numbers with $A\leqq a_j^{1/q}/b_j^{1/p}\leqq B$ $(j=1, 2, \cdots, n)$. Set $\gamma=B/A$. Then*

$$(8)\qquad 0\leqq\left[\left(\sum_{j=1}^{n} a_j^p\right)\Big/ \sum_{j=1}^{n} a_jb_j\right]^{1/p}-\left[\left(\sum_{j=1}^{n} a_jb_j\right)\Big/ \sum_{j=1}^{n} b_j^q\right]^{1/q} \leqq [\theta B^p+(1-\theta)A^p]^{1/p}-[\theta B^{-q}+(1-\theta)A^{-q}]^{-1/q},$$

where θ is the unique solution in $(0, 1)$ of

$$q(\gamma^p-1)[x(\gamma^p-1)+1]^{-1/q}+p(\gamma^{-q}-1)[x(\gamma^{-q}-1)+1]^{-(1/q)-1}=0.$$

Equality on the right in (8) holds if and only if there exists a subsequence $(k_1, k_2, \cdots, k_t)$ of $(1, 2, \cdots, n)$ such that $(\sum_{m=1}^{t} a_{k_m}b_{k_m})/\sum_{j=1}^{n} a_jb_j=\theta$, $a_{k_m}^{1/q}/b_{k_m}^{1/p}=B$ $(m=1, 2, \cdots, t)$, and $a_k^{1/q}/b_k^{1/p}=A$ for every k distinct from all k_m. Equality on the left in (8) holds if and only if all the ratios $a_j^{1/q}/b_j^{1/p}$ are equal.

Indeed, if we take in Theorem 1, $r=-q$, $s=p$, $q_j=a_jb_j/\sum_{k=1}^{n} a_kb_k$, $x_j=a_j^{1/q}/b_j^{1/p}$ $(j=1, 2, \cdots, n)$, we have by (3),

$$(9)\qquad 0\leqq M_p(x_1, x_2, \cdots, x_n)-M_{-q}(x_1, x_2, \cdots, x_n)\leqq \Delta.$$

Equality on the left holds if and only if all the x_j are equal. The difference in (9) equals the middle member of (8). The number Δ, by (4) and by the Remark in §5, is the right hand member of (8). The necessary and sufficient condition in Theorem 2 for equality on the right in (8) follows, too, from Theorem 1. The inequality on the left in (8) is, of course, just Hölder's inequality, and the condition given for equality there, is just the familiar condition for equality in Hölder's inequality.

9. A number of matrix inequalities follow from Theorems 1 and 2. For these inequalities the reader is referred to [**15**]. For corresponding Hilbert space in inequalities, see [**14**].

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