DECOMPOSITION OF PRODUCTS OF MODULAR REPRESENTATIONS[1]

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Let G be a cyclic group of order p^N with generator g, p a prime, and let KG be the group algebra over a field K of characteristic p. Green [1] and Srinivasan [3] gave formulas for the decomposition of tensor products of KG-modules into direct sums of indecomposables. We outline here an alternative procedure, based on the theory of elementary divisors, for obtaining these formulas.

For each r, $1 \leqq r \leqq p^N$, there is an indecomposable KG-module of dimension r. It affords a matrix representation $g \to V_r = E_r + H_r$, where E_r is the $r \times r$ identity matrix and H_r is the $r \times r$ matrix with ones along the superdiagonal and zeros elsewhere. The characteristic matrix $V_r - \lambda E_r$ has exactly one elementary divisor, $(1-\lambda)^r$. Thus the decomposition of a KG-module can be determined from knowledge of the elementary divisors of the matrix representation which it affords.

LEMMA ([2]). *The elementary divisors of* $V_m \otimes V_n - \lambda E_{mn}$ *are the same as those of* $M = [H_m + (1-\lambda)E_m]^n$.

Put $t = 1 - \lambda$. Expansion by the binomial theorem shows that M is an upper triangular matrix with (i, j) entry $a_{ij} = C(n, j-i)t^{n-j+i}$, $1 \leqq i, j \leqq m$, where the binomial coefficient $C(n, j-i)$ is to be regarded as an element of K.

To describe the elementary divisors, or what is the same, the invariant factors of M, we introduce the following notation. For $1 \leqq r \leqq m$, let $c(r)$ denote the largest integer l such that the submatrix of M consisting of the entries from rows 1 through r and columns l through m, has rank r. For example, $c(1)$ is the column index of the last nonzero entry of the first row of M.

We now indicate a procedure for finding the invariant factors of M. Subtract appropriate multiples of column $c(1)$ from columns preceding it so that all entries of the first row except $a_{1,c(1)}$ become 0. Subtract suitable multiples of the resulting first row from the rows below it so that all entries $(i, c(1))$, $2 \leqq i \leqq m$, become 0. If $c(1) < m$, repeat the process with columns $c(1)+1$ through m. These elementary operations transform M into a matrix

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$$\begin{bmatrix} 0 & D_1 \\ M_1 & 0 \end{bmatrix}$$

where D_1 is diagonal with each entry $a_{1,c(1)}$. It follows that the first $m-c(1)+1$ invariant factors of M are $t^{n-c(1)+1}$. It can be shown that each entry of M_1 is the determinant of an appropriate submatrix of M, which implies that either the upper right-hand entry of M_1 is nonzero or else M_1 has a triangular block of zeros in its upper right-hand corner. The procedure may then be repeated with M_1, leading to the following result.

THEOREM. *Let the sequence* $c(1), c(2), \cdots, c(m)$ *be of the form*

$$\begin{aligned} c(1) &= c(2) = \cdots = c(r_1 - 1) > c(r_1) \\ &= c(r_1 + 1) = \cdots > c(r_s) = 1 \end{aligned}$$

and set $r_0 = 1$. *Let*

$$e(i) = n - c(r_i) + r_i, \quad 0 \leqq i \leqq s.$$

Let $f_0 = m - c(r_0) + 1$ *and for* $1 \leqq i \leqq s$,

$$f_i = c(r_{i-1}) - c(r_i).$$

Then the invariant factors of M *are* $t^{e(i)}$ *with multiplicity* f_i, $0 \leqq i \leqq s$.

From this result one easily obtains the formulas of Green [1] used to show the semisimplicity of the representation algebra of KG.

REFERENCES

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