THE NOETHERIAN DIFFERENT OF PROJECTIVE ORDERS[1]

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1. Let K be a commutative ring and Λ a K-algebra (all rings will have identities and all modules will be unitary). Let $Z(\Lambda)$ be the center of Λ. Let $\phi_\Lambda: \Lambda \otimes_K \Lambda^0 \to \Lambda$ be given by $\phi_\Lambda(x \otimes y^0) = xy$. This is a homomorphism of left Λ^e-modules ($\Lambda^e = \Lambda \otimes_K \Lambda^0$). The set

$$N(\Lambda/K) = \{a \in Z(\Lambda): \exists f \in \operatorname{Hom}_{\Lambda^e}(\Lambda, \Lambda^e) \text{ with } \phi_\Lambda f = aI_\Lambda\}.$$

is an ideal in $Z(\Lambda)$ called the *Noetherian different* of Λ over K.

In this note we announce an extension of a result of D. G. Higman [2] to projective central orders over integrally closed integral domains. Details will appear in a paper of the same title in the *Journal für die reine und angewandte Mathematik* (*Crelle*).

It should be noted here that several authors have studied this ideal. See, for example, the papers listed in the bibliography.

2. Let K be an integrally closed integral domain with quotient field L. Let Λ be a K-order in a central simple L-algebra, Σ, which is projective as a K-module. Let $T: \Sigma \to L$ be the reduced trace from Σ to L. The hypothesis that K is integrally closed in L and that Λ is finitely generated as a K-module (it is an order) insures that $T(\Lambda) \subset K$. The complementary module, $C = C(\Lambda/K)$, and the Dedekind different, $D = D(\Lambda/K)$, are defined, as usual, as follows:

$$C = \{x \in \Sigma: T(x\Lambda) \subset K\},$$
$$D = \{x \in \Sigma: Cx \subset \Lambda\}.$$

Define the K-homomorphism t: $C \to \operatorname{Hom}_K(\Lambda, K)$ by $t(x)(y) = T(xy)$ for all $x \in C$, all $y \in \Lambda$.

PROPOSITION 1. *t is an isomorphism and* $t(D)(1) = T(D(\Lambda/K)) = N(\Lambda/K)$.

OUTLINE OF PROOF. It is shown first that each of the K-modules defined behaves nicely under localization at a maximal ideal of K. One

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such result, for instance, is $C(\Lambda \otimes_K K_M/K_M) = C(\Lambda/K) \otimes_K K_M$ where K_M is the ring K localized at a maximal ideal M.

This permits the study of the problem when Λ is free as a K-module. In this case an argument similar to Higman's in [2] yields the conclusion.

3. The following result permits the extension of Proposition 1 to a slightly more general class of ground rings K.

LEMMA 2. *Let Λ_i be K-algebras, $1 \leqq i \leqq n$. Then*

$$N\left(\bigoplus_{i=1}^{n} \Lambda_i/K\right) = \bigoplus_{i=1}^{n} N(\Lambda_i/K).$$

Proposition 1 may now be restated as follows:

THEOREM 3. *Let L be a field and Σ a separable L-algebra. Let K be an integrally closed subring of the center of Σ and Λ a K-order in Σ which is projective as a K-module. Let $T: \Sigma \rightarrow Z(\Sigma)$ be the direct sum of the corresponding reduced traces in the decomposition of Σ into central simple algebras. Define $C(\Lambda/K)$, $D(\Lambda/K)$ and t as before. Then t is an isomorphism and $T(D(\Lambda/K)) = N(\Lambda/K)$.*

4. When K is a Dedekind domain the theory of elementary ideals may be applied to the K-torsion module $C(\Lambda/K)/D(\Lambda/K)$ where Λ is a K-algebra as in §2. In fact:

PROPOSITION 4. *Let $H_1 \supseteq H_2 \supseteq \cdots \supseteq H_n$ be the elementary ideals of C/D. Then $H_1 = N(\Lambda/K)$.*

This result may be used to prove

PROPOSITION 5. *If K is a Dedekind domain with perfect residue class fields and Λ is a maximal order in a central simple L-algebra (L is the quotient field of K) then $N(\Lambda/K)$ is a square free product of those prime ideals containing $D(\Lambda/K) \cap K$.*

This last result suggests that the Noetherian different is a weaker invariant than the Dedekind different, or the discriminant, in the case of central K-algebras, since it does not reflect the amount of ramification between K and Λ.

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