A COUNTEREXAMPLE ON RELATIVE REGULAR NEIGHBORHOODS

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Hudson and Zeeman defined the concept of a relative regular neighborhood in [1], and gave an existence theorem and two uniqueness theorems; the purpose of this note is to show that the uniqueness theorems are false. A corrected version of these theorems has been announced by L. S. Husch and will appear later. For 3-manifolds, however, the corrected version is equivalent to the original.

For general terminology and definitions, see Zeeman [2]. Suppose K, L are subcomplexes of some complex J. We say that K is *link collapsible* on L if $lk(A, \mathrm{Cl}(K-L))$ is collapsible for any simplex A of $\mathrm{Cl}(K-L)\cap L$. If X and Y are compact polyhedra in a polyhedral manifold M, we say that X is link collapsible on Y if there is a triangulation K, L of X, Y such that K is link collapsible on L. For example, it is easy to see that a manifold is always link collapsible on any subpolyhedron of its boundary. Let X, Y, N be compact polyhedra in M. We say that N is a *regular neighborhood of* X mod Y *in* M if

(1) N is an m-manifold ($m=\dim M$),

(2) N is a topological neighborhood of $X-Y$ in M and

$$N \cap Y = N \cap Y = \mathrm{Cl}(X - Y) \cap Y,$$

(3) N collapses to $\mathrm{Cl}(X-Y)$.

The uniqueness theorems given by Hudson and Zeeman say, among other things, that *any two regular neighborhoods of* X mod Y *in* M *are homeomorphic keeping* $\mathrm{Cl}(X-Y)$ *fixed*, provided X is link collapsible on Y.

Let (B^3, B^1) be a knotted 3, 1 ball pair in E^3 and let $B^4=a * B^3$ and $B^2=a * B^1$ where $a=(0, 0, 0, 1)\in E^4$ and $*$ denotes join. The 4, 2 ball pair (B^4, B^2) is locally knotted at the vertex a and hence is knotted. However it is easy to see that B^2 is unknotted in E^4. Let $h: E^4\to E^4$ be an onto piecewise linear homeomorphism such that $h(B^2)=\Delta$ is a 2-simplex. B^4 collapses cone-wise to B^2, so that $h(B^4)$ collapses to $h(B^2)=\Delta$. Also $\dot{\Delta}\subset h(\dot{B}^4)$ and $\mathring{\Delta}\subset h(\mathring{B}^4)$, so that $h(B^4)$ is a regular neighborhood of Δ mod $\dot{\Delta}$ in E^4. Let Σ be the 2-fold suspension of Δ in E^4; then Σ is a regular neighborhood of Δ mod $\dot{\Delta}$ in E^4.

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If the above uniqueness theorem were true, there would be a homeomorphism carrying $h((B^4), \Delta)$ onto (Σ, Δ) which implies that $(h(B^4), \Delta) = (h(B^4), h(B^2))$ is an unknotted ball pair. This is clearly false since it is a homeomorph of the knotted ball pair (B^4, B^2). Clearly this type of argument can be carried out in E^n for every $n \geqq 4$, so that the uniqueness theorem is false for all dimensions greater than 3; as remarked above, the theorem is true for 3-manifolds.

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References

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2. E. C. Zeeman, *Seminar on combinatorial topology*, Inst. Hautes Etudes Sci. Publ. Math., Paris, 1963. (mimeographed notes)

3. L. S. Husch, *On relative regular neighborhoods. Preliminary report*, Abstract 66T-212, Notices Amer. Math. Soc. **13** (1966), 386.

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