ON AUTOMORPHISMS OF ALGEBRAIC GROUPS[1]

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Automorphisms of Lie algebras over fields of characteristic 0 have been investigated by Borel-Mostow, Jacobson, and Patterson (see [1], [4], [5]). This note describes some results in [6] of related investigations of automorphisms of algebraic groups over fields of arbitrary characteristic.

In the following discussions, G is a connected linear algebraic group over an algebraically closed field of characteristic 0 or p. σ and τ are (birational) automorphisms of G. The connected component of the identity of the group of fixed points of σ is denoted by $F_G(\sigma)$.

σ acts on the algebra $R(G)$ of representative functions of G. σ is said to be algebraic if the orbit of each element of $R(G)$ under the cyclic group generated by σ spans a finite dimensional subspace of $R(G)$.

An algebraic automorphism σ is said to be semisimple (unipotent) if the induced transformation on $R(G)$ is semisimple (unipotent). If σ is algebraic, σ has a unique decomposition $\sigma=\sigma_s\sigma_u$ where σ_s, σ_u are commuting algebraic automorphisms which are respectively semisimple and unipotent. If G is semisimple, every (birational) automorphism of G is algebraic (see [2], §17-07).

If σ is an algebraic automorphism of G, then there is a linear algebraic group K containing G as a closed normal subgroup and an element s in K such that σ is the restriction to G of the inner automorphism Ad s. If σ is a semisimple (unipotent) algebraic automorphism of G, s may be taken to be semisimple (unipotent); and such a σ may be regarded as a semisimple (unipotent) element of K by identifying σ with such an s. On the other hand, elements σ, τ of K are sometimes regarded as automorphisms of G in the following discussions. (The above follows easily from results in [3].)

THEOREM 1. *Let σ and τ be semisimple elements of a linear algebraic group K containing G as a closed normal subgroup. Suppose that $\sigma G=\tau G$. Let H be a Cartan subgroup of $F_G(\sigma)$, L a Cartan subgroup of $F_G(\tau)$. Then there exists an element g in G such that $g^{-1}Hg=L$ and $g^{-1}\sigma Hg=\tau L$.*

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SKETCH OF PROOF. By regarding G as a transformation group on σG (an element g in G sending σx in σG into $g^{-1}\sigma xg$ in σG), it can be shown that the set $\{g^{-1}\sigma xg \mid g \in G,\ x \in 0_H\}$ is a dense épais subset of σG, provided that 0_H is a dense épais subset of H. Such an 0_H can be chosen such that for h in 0_H, H is the connected centralizer in G of σh_s where h_s is the semisimple part of h. (This and the preceding fact are established by applications of a transformation group formulation of Lemma 5, §6–11 of [2].) There is a similar 0_L for (τ, L). Since two dense épais subsets of σG have nonempty intersection, there exist g_1, g_2 in G, h in 0_H and l in 0_L such that $g_1^{-1}\sigma hg_1 = g_2^{-1}\tau lg_2$. Thus $g_1^{-1}\sigma h_s g_1 = g_2^{-1}\tau l_s g_2$ and taking connected centralizers in G, $g_1^{-1}Hg_1 = g_2^{-1}Lg_2$. The assertions of the theorem follow immediately.

THEOREM 2. *Suppose that G is semisimple. Then if τ keeps stable a maximal torus T and a Borel subgroup B containing T, $F_T(\tau)$ contains a regular element of G and is a Cartan subgroup of $F_G(\tau)$.*

PROOF. The cyclic group A generated by τ acts in the group T^* of rational characters of T and keeps stable the subset S of fundamental roots of T with respect to B. Let m be the index in T^* of the subgroup generated by S. Assume that $\dim G > 0$ and let t be an element of T of finite order such that $\alpha(t) = \beta(t)$ whenever α and β are elements of S which lie in the same orbit under A. Then $\alpha(\tau(t)) = \alpha(t)$ for α in S. Thus $\chi^m(\tau(t)) = \chi^m(t)$ for χ in T^*. Thus $\chi(\tau(t^m)) = \chi(t^m)$ for χ in T^* and $\tau(t^m) = t^m$ since T^* separates points. The order of t (and hence of t^m) can be taken to be arbitrarily large. Thus $\dim F_T(\tau) \geqq 1$. Let $T_1 = F_T(\tau)$ and let G_1 be the connected centralizer of T_1. G_1 is reductive with maximal torus T and Borel subgroup $B \cap G_1$ (see [2]). G_1, T, and $B \cap G_1$ are τ-stable. Thus if $\dim G_1^{(1)} > 0$, an application of the above argument shows that $\dim F_{T_2}(\tau) \geqq 1$ where $T_2 = T \cap G_1^{(1)}$. This is impossible since $F_{T_2}(\tau) \subseteq T_1 \cap G_1^{(1)}$ and $T_1 \cap G_1^{(1)}$ is finite. Thus $\dim G_1^{(1)} = 0$. Thus $G_1 = T$ and $F_T(\tau)$ contains a regular element of G. It now is immediate that $F_T(\tau)$ is a Cartan subgroup of $F_G(\tau)$.

THEOREM 3. *Let G be semisimple and let σ be a semisimple (algebraic) automorphism of G. Let T be a maximal torus of G, B a Borel subgroup of G containing T. Then there exists g in G such that $g^{-1}Tg$ and $g^{-1}Bg$ are stable under σ. $F_G(\sigma)$ contains a regular element of G.*

PROOF. Regarding σ as a semisimple element of an algebraic linear group K containing G as a closed normal subgroup, choose a semisimple element τ of σG such that Ad τ keeps stable T and B (possible by the conjugacy of maximal tori and Borel subgroups under inner automorphisms). Then $\sigma G = \tau G$ and $F_T(\tau)$ is a Cartan subgroup of

$F_G(\tau)$ (Theorem 2). Thus letting H be a Cartan subgroup of $F_G(\sigma)$, there exists g in G such that $g^{-1}F_T(\tau)g = H$ and $g^{-1}\tau F_T(\tau)g$ contains σ (Theorem 1). For such a g, $g^{-1}Tg$, $g^{-1}Bg$ are σ-stable since T, B are τ-stable. An application of Theorem 2 now shows that $F_G(\sigma)$ contains a regular element of G.

Theorem 3, along with Theorem 2 and the methods used in its proof, can be used to compute the rank of $F_G(\sigma)$ where σ is a semisimple algebraic automorphism of a semisimple algebraic group G (the rank of $F_G(\sigma)$ corresponds to the index of "σG" in [4]).

A straightforward consequence of the preceding theorem is

COROLLARY 4. *Let σ be a semisimple algebraic automorphism of G. Then*

(1) *σ keeps stable a Borel subgroup of G;*

(2) *σ keeps stable a maximal torus of G;*

(3) *the centralizer in G of a maximal torus in $F_G(\sigma)$ is solvable.*

R. Steinberg has independently proved part (1) of Corollary 4, using methods which require only that one assume that σ be a birational automorphism of G.

The proofs of the following two theorems will appear in a later paper.

THEOREM 5. *If σ has only finitely many fixed points, then G is solvable.*

THEOREM 6. *Suppose that σ has finite order n and that σ has only finitely many fixed points in G. Then σ keeps stable precisely one maximal torus T_σ of G, and the fixed points of σ are elements of T_σ whose orders divide n. If n is prime, G is nilpotent.*

REFERENCES

1. A. Borel and G. D. Mostow, *On semi-simple automorphisms of Lie algebras*, Ann. Math. **61** (1955), 389–504.

2. C. Chevalley, *Classification des groupes de Lie algêbriques*, Sêminaire C. Chevalley, Vols. I, II, École Norm. Sup. Paris, 1958.

3. G. Hochschild and G. D. Mostow, *Representations and representative functions of Lie groups*, Ann. Math. **66** (1957), 495–452.

4. N. Jacobson, *A note on automorphisms of Lie algebras*, Pacific J. Math. **12** (1962), 303–315.

5. E. M. Patterson, *On regular automorphisms of certain classes of rings*, Quart. J. Math. **12** (1961), 127–133.

6. D. J. Winter, *On automorphisms of Lie algebras and algebraic groups*, Doctoral Dissertation, Yale University, New Haven, Conn., 1965.

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