THE STRUCTURE OF TOPOLOGICAL SEMIGROUPS—REVISITED

BY PAUL S. MOSTERT

To Professor M. L. MacQueen on his retirement, June, 1966, after forty-three years of service to Southwestern at Memphis.

Twelve years ago, A. D. Wallace, who might well be called the founder of the theory of compact topological semigroups, in a talk entitled *The structure of topological semigroups* [29], addressed the Society in this same capacity for the purpose of stating that there was no such thing as a structure theorem. "There does not exist at this time," he said, "any corpus of information to which the title 'structure of topological semigroups' is in any fashion applicable. Whether such a body of theorems will ever exist is a matter for the future . . ." He went on to say that, contrary to the implications of the title, he would *not* talk about a small number of large theorems, but rather about a large number of small ones, because that was all that the then infant theory had to offer. Today, it is possible to carry through with the implications of his title, and so, in honor of his 60th birthday this past August 21st, 1965, I am going to talk about a small number of large theorems. In fact, primarily I am going to talk about just a single theorem—a structure theorem—which K. H. Hofmann and I in our forthcoming book [10] call the *Second fundamental theorem of compact semigroups.* (Some open questions will also be discussed, and with these, a few additional results stated.)

But in 1953, the theory of topological semigroups was hardly three years old. To be sure, there was the forgotten paper of Eilenberg's [5] in 1937, and a paper of Iwasawa's (in Japanese) [18] in 1948, but it was not until 1950 that any concerted effort in the area began—more or less simultaneously from several directions: there was the short, but very useful note by Gleason [8] and papers by Peck [25], by Gelbaum, Kalish and Olmstead [7], and by Numakura [23] a year later. But it was with the publication of papers by Numakura [24], Wallace [27], [28] in 1952 and 1953, and the dissertation of Koch [19], 1953, that the theory really began to move. By 1956, the

An address delivered before the Lexington meeting of the Society on November 13, 1965, by invitation of the Committee to Select Hour Speakers for Southeastern Section Meetings.

main elements of the First Fundamental Theorem, which I shall describe presently, had been obtained. Due largely to Wallace himself [30] it is, except for a few frills that Hofmann and I have added, now well known to workers in the area.

In the early years, there was a great deal of groping about for the right techniques, and the beautiful structure theory for compact groups offered tantalizing bait for many deceiving conjectures. With his numerous ingenious examples, Hunter, see e.g. [14], [16], [17] set about destroying many of these, and it soon became obvious that neither the techniques nor the results of group theory could be extended to the realm of compact semigroups. New ideas about how these objects were to be described and new techniques—techniques peculiar to the area—were needed for obtaining this description. A recognition of what the correct building blocks should be, and a thorough knowledge of these had to be found. These new techniques have now begun to make their appearance, and enough of the basic building blocks can now be described to give us some rather solid information—at least about certain categories of compact semigroups.

However, let us begin now with some definitions so that we can be more explicit.

A *semigroup* is a nonempty Hausdorff space with a (jointly) continuous and associative multiplication. The results we are to describe apply only to compact semigroups, and in fact it is for the most part with the category of compact connected semigroups with identity that we wish to pursue our study.

Let S be a semigroup. An *identity* is an element $1\in S$ such that $1x=x1=x$ for all $x\in S$. The *group of units* (or *maximal subgroup*) of S is the set $H(1)=\{x: xy=yx=1 \text{ for some } y\in S\}$. In a compact semigroup, $H(1)$ is a compact group and thus is a quite comfortably "known" object—at least from the point of view of one working in topological semigroups. If $e^2=e\in S$, then $H(e)$ the group of units of eSe, is the *maximal group* of e.

A *left* (*right*) *ideal* of S is a set $I\subset S$ such that $SI\subset I$ (resp., $IS\subset I$). An *ideal* is simultaneously a left and right ideal. Every compact semigroup has a unique minimal ideal $M(S)$ and it is closed. If was recognized fairly early by Wallace [30] that $M(S)$ satisfied the hypotheses of a completely simple semigroup, the algebraic structure of which had previously been worked out by Suschkewitsch and Rees (see [4]). The topological properties were then easy consequences. Wallace also observed that the Čech global cohomological properties of S were concentrated in any maximal group in $M(S)$. Thus, it is possible to describe $M(S)$ in the following terms:

DEFINITION 1. Let X be a topological space. Define two multiplications on X as follows:

(a) X has *left zero multiplication* if $xy=x$ for all $x, y \in X$.

(b) X has *right zero multiblication* if $xy=y$ for all $x, y \in X$.

Let X and Y be compact spaces and G a compact group. Let

$$\sigma: Y \times X \to G$$

be a continuous function. Denote $\sigma(y, x)=[y, x]$. Define the *Rees product* of X, G, Y as the space $X \times G \times Y$ with multiplication defined by

$$(x, g, y)(x', g', y') = (x, g[y, x']g', y').$$

(Notice that if G and Y are degenerate, this amounts to left zero multiplication on X. A similar statement applies to Y.) This results in a continuous associative multiplication on $X \times G \times Y$ which we denote by $[X, G, Y]_\sigma$. Any compact semigroup isomorphic to a Rees product of this form we call a *paragroup*.

We can now state the

FIRST FUNDAMENTAL THEOREM. *Let S be a compact semigroup. Then there is a unique minimal ideal $M(S)$ and it is a paragroup $[X, G, Y]_\sigma$. The sets $X \times G \times y$ (resp., $x \times G \times Y$) are the minimal left (resp. right) ideals of $M(S)$ and hence of S. Moreover, there is a sequence of surjective morphisms*

$$S = S_1 \xrightarrow{f_1} S_2 \xrightarrow{f_2} S_3 \xrightarrow{f_3} S_4$$

such that $f_i | (S_i \backslash M(S_i))$ is a homeomorphism onto $S_{i+1} \backslash M(S_{i+1})$, $i=1, 2, 3$, and

(1) *$M(S_2)$ is isomorphic to $[X, e, Y]$ and for each $(x, e, y) \in M(S_2)$, $f_1^{-1}(x, e, y)=(x, G, y)$, which is a maximal subgroup of $M(S)$, where e is the identity of G.*

(2) *$M(S_3)$ is isomorphic to X with left zero multiplication, and $f_2^{-1}(x)$ is a minimal right ideal of $M(S)$.*

(3) *$M(S_4)$ is a zero for S_4.*

If, moreover, S is connected and has an identity, then the inclusion map $i: (x, G, y) \to S$ induces an isomorphism $i^*: \check{H}^*(S) \to \check{H}^*(G)$ of cohomology groups (relative to any coefficient group). The sets X and Y are acyclic relative to Čech cohomology. (Relative to singular theory, this need not be so.)

Thus, we can feel just about as confident in our knowledge of the structure of the minimal ideal of a compact connected semigroup with identity as in the knowledge of its group of units. But these are only settled villages at either extreme of a huge forest. We want to gain some knowledge of the mysteries of the interior. A little reflection and a few examples will soon dispel any delusions about the possibilities of an adequate charting of this wilderness. However, there *is* a path through the wilderness joining the two ends.

DEFINITION 2. Let T be a compact connected semigroup with identity 1. If T contains no proper compact connected subsemigroup meeting $M(T)$ and containing 1, we call T *irreducible.*

Zorn's lemma tells us

LEMMA 1. *Let S be a compact connected semigroup with identity* 1. *Then there is an irreducible semigroup $T\subset S$ such that $1\in T$ and $M(S)\cap T\neq\emptyset$.*

Now we have planned a trail through the forest. Can we at least describe the trail? That is the essential content of the Second Fundamental Theorem and the results leading thereto.

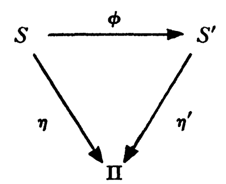
We proceed to describe these semigroups. But first, let us agree on when we have an adequate description.

A preferred method of describing the structure of objects in many categories is to find isomorphic, or at least sufficiently many homomorphic, representations of the object into objects whose structure is rich enough to be more amenable to investigation than the original object. This is true, in particular, of the categories of connected compact and locally compact groups, commutative Banach algebras, Boolean rings, and Lie algebras, and in fact in some sense to abelian categories via the Mitchell-Freyd full embedding theorem. But there can be no direct analogue to the representation theory for compact groups as semigroups of endomorphisms of topological vector spaces because of the existence of too many idempotents in most compact semigroups. Probably certain subcategories of semigroups can be treated successfully by means of sufficiently many linear or affine representations, however.

Our approach is rather by constructive methods. The technique is to produce first a few basic building blocks—semigroups which are easily described via simple formulae. From these building blocks, we construct more complicated structures. We will thus consider the structure of the objects in a category as *known* if we have constructive rules which allow us to decompose the structure into the simplest building blocks and then compose them again to obtain a topologically and algebraically isomorphic structure. We must, in this process, allow a certain restrained use of homomorphic images—or *surmorphism*.[1] To illustrate what is meant by this statement we shall consider an example. But first we need some definitions.

DEFINITION 3. Let S be a compact semigroup. We shall let $\mathcal{H}$ denote the set of pairs (x, y) in $S \times S$ such that x and y generate the same principal left and principal right ideals. That is, $x \cup xS = y \cup yS$ and $x \cup Sx = y \cup Sy$. $\mathcal{H}$ is a closed equivalence relation. Let $\eta: S \to S/\mathcal{H}$ denote the natural projection.[2]

EXAMPLE 1. Let $\mathcal{C}$ denote the category whose objects are compact semigroups such that $\mathcal{H}$ is a congruence relation and $S/\mathcal{H}$ is isomorphic on $\Pi = [0, 1]$ relative to ordinary multiplication of real numbers, and whose morphisms $\phi: S \to S'$ are surjective morphisms which make the diagram



commute. If we succeed in describing one object S in this category, we shall consider all objects S' in the category known for which there is a morphism $\phi: S \to S'$ in the category. The limits imposed on the morphisms are sufficient to justify this approach.

To further illustrate this approach and to build up our store of building blocks, let Π_r, $0 \leq r \leq 1$, denote the interval $[r, 1]$ with multiplication given by $x \cdot y = \max\{xy, r\}$. (This is the semigroup obtained from $\Pi = \Pi_0$ by identifying the ideal $[0, r]$ to a point.) It is not difficult to show that Π_r is isomorphic to $\Pi_{1/2}$ for $0 > r > 1$ and is characterized by the property that it is a totally ordered compact connected semigroup with identity and zero as end points, and every element (except the identity) is nilpotent.

---

[1] We use the word *surmorphism* to mean "morphism onto." One would expect the more euphoneous word "epimorphism" to be used in this context, but alas, an epimorphism in the category of compact semigroups, or even in the more restricted category of compact connected semigroups, need not be onto.

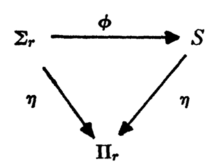
[2] The relation $\mathcal{H}$ and the equivalence relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{D}$ determined similarly by the principal left, respectively, right, respectively, two sided ideals play a role in the theory of compact semigroups similar to that played by the taking of quotient spaces in the theory of topological groups. Notice that $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$.

A *solenoidal semigroup* is a compact semigroup with a dense one-parameter subsemigroup (i.e. a dense homomorphic image of the multiplicative semigroup (0, 1)). A solenoidal semigroup is either a solenoidal group (i.e., a compact group with a dense one-parameter subgroup [9]), or an irreducible semigroup such that $S/\mathcal{H}\approx\Pi_r$ for some $0\leqq r<1$. A further characterization, is as follows: let A be a solenoidal group and f: $(0, 1]\to A$ be a dense one-parameter subsemigroup. Then $\Pi\times A$ is a compact semigroup relative to coordinatewise multiplication. Let S be the subsemigroup given by

$$S = \{(x, f(x)) : x \in (0, 1]\} \cup \{0\} \times A.$$

Then S is a compact semigroup.

Now if $0\leqq r<1$, the equivalence relation R which collapses the sets $[0, r]\times\{a\}$ to a point for each $a\in A$ is a congruence relation. Let $S_r=S/R$. Then S and S_r are solenoidal semigroups and any solenoidal semigroup is either a solenoidal group, or is obtained in this way. If A is the universal solenoidal group, that is the Baire compactification $(\boldsymbol{R}_d)^\wedge$ of the reals, then the resulting semigroups are denoted by Σ and Σ_r (relative to $f=\hat{\imath}$, where i: $\boldsymbol{R}_d\to\boldsymbol{R}$ is the identity map). Every solenoidal semigroup S is a surmorphic image $\phi: \Sigma_r\to S$ in such a way that



is a commutative diagram.

We now have at our disposal a number of important building blocks. One further form is required before we are ready to tear down and reconstruct irreducible semigroups. The basic building blocks of irreducible semigroups are the *cylindrical* semigroups. For the purposes of treating irreducible semigroups, it is enough to consider cylindrical semigroups with connected abelian groups. But we wish to do a little more. Cylindrical semigroups are formed as follows: Let H be a compact group. A semigroup S is said to be *cylindrical* if it is a surmorphic image of a semigroup of the form $\Sigma\times H$ where H is a compact group. This is actually a great deal stronger statement than it may at first appear. We know, for example, that if S is not a group, $S/\mathcal{H}$ is then isomorphic to either Π or $\Pi_{1/2}$ and that S can in fact be obtained from $\Sigma\times H$ or from $\Sigma\times H$ by factoring a totally ordered collection of closed, normal subgroups $\{H_x: x\in(0, 1])$ of H and

$H_0 \subset (R_d)^\wedge \times H$ such that

(a) $x < y$ implies $H_y \subset H_x$,

(b) $H_x = \bigcap \{H_y : x > y\}$,

(c) $1 \times \bigcup \{H_x : 0 \leqq x \leqq 1\} \subset H_0$,

where in the second case we also specify that $H_x = H_r$ if $x \leqq r$. In fact, we can again take the attitude that a cylindrical semigroup is a sufficiently familiar object to use as a building block in our project.

Now, according to our promise, we are ready to describe the reconstruction of an irreducible semigroup. Having described so far only semigroups which are abelian—or are nearly so—we should perhaps at this point give out a part of the secret.

THEOREM 1. *An irreducible semigroup is abelian.*

The proof of this fact was a major step in the unveiling of the mysteries of irreducible semigroups. Its proof, the skeleton of which utilizes a generalization of an argument of Koch's [**20**], is based on the following two important results, the first of which is due to Mostert and Shields [**22**]. (The main argument for the second was given to Hofmann and me by A. Borel and is a generalization of a result of Conner's [**6**]. Our original proof was valid only for groups with the property that for some prime p, the collection of cyclic groups of order p^n for $n = 1, 2, \cdots$ forms a dense subset of the given group.)

THEOREM 1a (MOSTERT-SHIELDS). *Let S be a compact connected semigroup with identity 1_S and suppose there is a neighborhood of 1_S in which there is no other idempotent. If $S \neq H(1)$, then there is a morphism $\phi: \Sigma \rightarrow S$ such that $\phi(1_\Pi) = 1_S$ and $\phi(M(\Sigma)) \not\subset H(1_S)$.*

THEOREM 1b. *Let G be a compact connected abelian group acting as a transformation group on a compact acyclic space X. Then the fixed point set of G is acyclic.*

(A space is said to be *acyclic* if its Čech cohomology ring is isomorphic to that of a point.)

The proof of Theorem 1 can now be outlined. Let us assume for the moment that S has a zero, or at least that all groups are trivial. This we may do because of the First Fundamental Theorem. We shall show that a compact connected semigroup S with identity contains a connected abelian semigroup containing the identity and meeting the minimal ideal. We may do this for S_2 (or S_4) and then lift back to S. Now let $\mathfrak{U}$ be an entourage of the uniform structure of S, and let $\mathcal{C}_{\mathfrak{U}}$ be the collection of all compact abelian subsemigroups $T\subset S$ such that $1\in T$, $M(T)$ is connected, and T is $\mathfrak{U}$-connected. (That is, if $x, y\in T$, there is a finite collection of points $x=x_0, x_1, \cdots, x_n=y$ in T such that $(x_{i-1}, x_i)\in\mathfrak{U}$, $i=1, \cdots, n$.) Order $\mathcal{C}_{\mathfrak{U}}$ by inclusion. It is not difficult to show that it is inductive. Let $T_{\mathfrak{U}}$ be a maximal element in $\mathcal{C}_{\mathfrak{U}}$. We want to show that $T_{\mathfrak{U}}\cap M(S)\neq\emptyset$. Suppose on the contrary that $T_{\mathfrak{U}}$ does not meet $M(S)$. The hypotheses on $T_{\mathfrak{U}}$ imply, by the First Fundamental Theorem, that $M(T_{\mathfrak{U}})$ is an abelian group. Let e denote its idempotent. Then $M(T_{\mathfrak{U}})$ acts on the acyclic space eSe under $s\rightarrow g^{-1}sg$ and leaves e and the zero of S (or some point in the acyclic space $M(eSe)$) fixed. Since the fixed point set of $M(T_{\mathfrak{U}})$ is connected by Theorem 1b, the centralizer Z of $M(T_{\mathfrak{U}})$ is connected, contains e, and meets $M(S)$. If $\mathfrak{U}(e)\cap Z$ contains an idempotent f different from e, then $T_{\mathfrak{U}}\cup fM(T_{\mathfrak{U}})$ is a properly larger semigroup in $\mathcal{C}_{\Sigma}$. If $\mathfrak{U}(e)\cap Z$ contains no idempotent except e, then by Theorem 1a there is a morphism $\phi:\Sigma\rightarrow eSe\cap Z$ such that $\phi(1_{\mathfrak{U}})=e$ and $\phi(M(\Sigma))\not\subset\mathfrak{U}(e)$. Then $T_{\mathfrak{U}}\cup M(T_{\mathfrak{U}})\phi(\Sigma)$ is again a properly larger semigroup in $\mathcal{C}_{\mathfrak{U}}$. Thus, $T_{\mathfrak{U}}$ does indeed meet $M(S)$. Now a limit over all entourages of the uniform structure yields a compact connected abelian semigroup T such that $1\in T$ and $T\cap M(S)\neq\emptyset$. (The full details of this will appear in [**10**].)

Once we are aware that irreducible semigroups are abelian, life becomes much easier. One can list a number of properties—such as (1) that the group of units is trivial, and (2) that $S/\mathcal{H}$ is a connected totally ordered semigroup with zero and identity as endpoints (i.e., an *I-semigroup*) and these semigroups are all long known through the work of Clifford [**24**] and Mostert and Shields [**21**]. These are properties of abelian irreducible semigroups proved by, in the first case, Hunter [**16**], and in the second case, Hunter and Rothman [**17**]. Moreover, examples by Hunter and Rothman [**19**] and by Hunter [**14**], [**16**] gave a great deal of insight into what one must look for in a structure theorem. Actually, the construction technique I shall describe below has much wider applications, and so I shall in this instance not merely limit myself to the abelian case. The semigroups constructed here occupy a valuable place in the theory of compact connected semigroups, as we shall see by Theorems 1 and 2 below. Since they are formed by the chaining of well known objects—namely the cylindrical semigroups—like ornaments along a chain of idempotents, Hofmann and I have for this reason given the semigroup resulting from such a construction the name *hormos*, from ὅρμος, meaning "ornamental chain." We now proceed with the construction:

THE HORMOS. The hormos is constructed from a collection (X, S_x, m_{xy}), where

(a) X is a totally ordered compact set.

We shall denote the minimal element of X by 0. If $a, b \in X$, $]a, b[= \{x \in X: a < x < b\}$. Intervals $[a, b]$, $[a, b[$, etc. are similarly defined. Define $X' = \{x \in X:]y, x[= \emptyset$ for some $y < x\}$.

(b) S_x, for each $x \in X$, is a cylindrical semigroup.

Let H_x denote the group of units of S_x and $M_x = M(S_x)$. Let 1_x denote the identity of S_x and e_x the identity of M_x. We require that $x \notin X'$, iff $H_x = S_x = M_x$.

(c) For each pair $x, y \in X$, $x \leqq y$, $m_{xy}: S_y \to S_x$ is a homomorphism satisfying the following properties:

(i) m_{xx} is the identity on S_x.

(ii) If $x < y$, then $m_{xy}(S_y) \subset H_x$.

(iii) If $x < y < z$, then $m_{xy} m_{yz} = m_{xz}$.

(iv) $m_{xy} | M_y$ is an injection if $x = y'$, $y \in X'$, where for $y \in X'$, $y' = \text{l.u.b.}\{z < y\}$.

(v) The map $\phi_x: H_x \to \pi\{H_y: y < x\}$ defined by $\phi_x(g) = (m_{yx}(g))_{y<x}$, for $0 < x \notin X'$, is an isomorphism onto the projective limit $\lim_{\leftarrow}\{H_y, m_{yz}, y < z < x\}$. Let S' denote the disjoint union of the S_x, $x \in X$, let $p: X' \to X$ be defined by $p(s) = x$ if $s \in S_x$, and define multiplication on S' by

$$s \cdot t = \begin{cases} st & \text{if } s, t \in S_{p(s)}, \\ s \cdot m_{p(s)p(t)}(t) & \text{if } p(s) < p(t), \\ m_{p(t)p(s)}(s) \cdot t & \text{if } p(t) < p(s). \end{cases}$$

For example, if the cylindrical semigroup $S_{p(s)}$ and $S_{p(t)}$ are in the relationship shown in Figure 1, multiplication represents a rotation of s in its H-class by the image of t under the map $m_{p(s)p(t)}$. (It is not difficult to show that the H-class of an element s is just $s \cdot H_{p(s)}$.)

Now on S' we define a basis for a topology as follows: Let $\mathfrak{B}$ be a basis of open intervals on X, and let $\mathfrak{B}_x$ be the set of open sets in $S_x \backslash H_x$, $x \in X$. For $U \in \mathfrak{B}$, let $u = \text{g.l.b. } U$, and let V be open in S_u with $H_u \cap V \neq \emptyset$. Define

$$W(U, V) = p^{-1}(U) \cap \{s: m_{up(s)}(s) \in V\}.$$

Then $\bigcup\{\mathfrak{B}_x: x \in X\} \cup \{W(U, V): U \in \mathfrak{B}, V \text{ open in } S_u\}$ is a basis for a compact Hausdorff topology in S'.

We are not quite finished. We want a semigroup S such that $S/\mathcal{H}$ is an I-semigroup—that is, with $S/\mathcal{H}$ a *connected* totally ordered space.

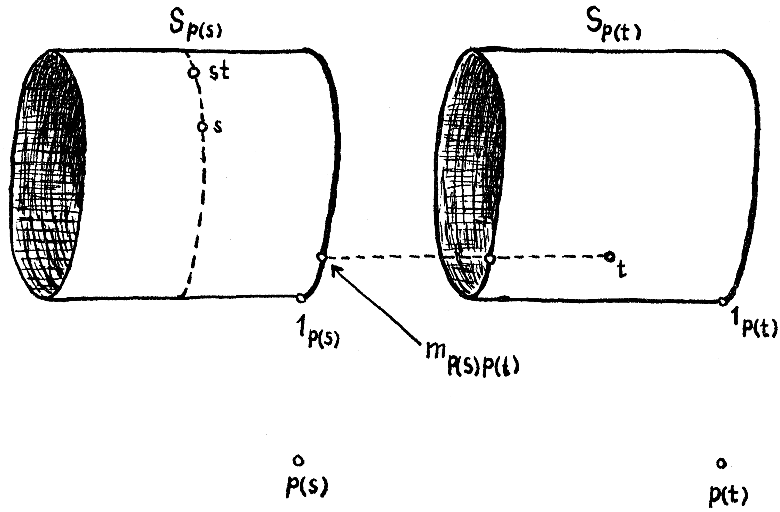


FIGURE 1

At present, $S'/\mathcal{H}$ is ordered, but is not connected unless X is connected. We define an equivalence relation R on S' by $R(s)=\{s\}$ if $p(s)\notin X'$ or $p(s)\neq x'$ for any $x\in X'$, and $R(s')=\{m_{x'x}(s), s\}$ if $s'=s$ or $s'=m_{x'x}(s)$ for some s with $p(s)=x\in X'$. R is in fact a closed congruence relation. Define $S=\mathrm{Horm}(X, S_x, m_{xy})=S'/R$. This semigroup is a compact semigroup and $S/\mathcal{H}$ is an I-semigroup. (An I-semigroup is a hormos also—but a quite trivial one. In this case, for each $x\in X$, $S_x=\Pi$, for some $r\leqq 1$ depending on x.) We are able to prove

THEOREM 2. *Let S be a compact semigroup. Then $S=\mathrm{Horm}(X, S_x, m_{xy})$ for some collection (X, S_x, m_{xy}) satisfying the conditions* (a)–(c) *iff $S/\mathcal{H}$ is an I-semigroup.*

In the case of irreducible semigroups, we are going to be able to say much more, but first let us illustrate this rather formidable sounding construction with a few simple examples.

Let $X=\{-1, 0, 1\}$ with the natural order. Let S_1 be the cylindrical semigroup given previously—i.e. a one-parameter semigroup winding down on the circle group. Let S_0 be the unit disk in the complex plane under ordinary multiplication of complex numbers, and let $S_{-1}=0$, a one point semigroup. The map $m_{01}\colon S_1\to S_0$ is given by $m_{01}(r, \exp[2\pi ix])=\exp[2\pi ix]$, $m_{-10}=0$, and m_{xx} the identity. We illustrate this in Figure 2.

The interval at the right represents $S/\mathcal{H}$. There are the 3 idempotents 1, 0, -1 (relative to multiplication $xy=\min(x, y)$) and between the idempotents, multiplication is isomorphic to multiplication on the ordinary unit interval. The semigroup S so constructed is, in this case, also irreducible.

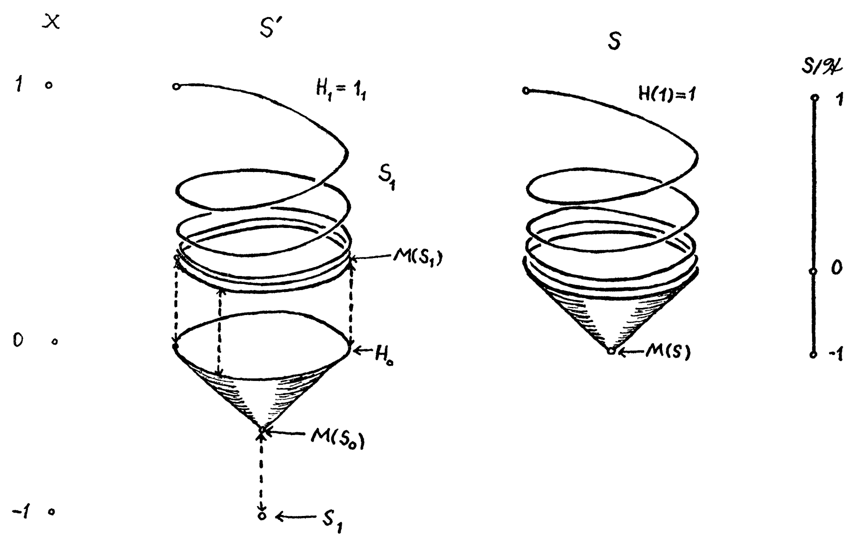


FIGURE 2

Let us illustrate the construction now with a more complicated, and correspondingly more interesting example. If we take the product of the semigroup used for S_1 in the previous example and the circle group T, and then collapse each set $(0, a)\times T$ to a point, we obtain the semigroup illustrated in Figure 3.

Let $X=\{1/2^n, 1-1/2^n: n=0, 1, \cdots\}$, $S_0=S_1=\{0\}$, a one point semigroup, and let S_x be isomorphic to the semigroup given above in Figure 3 for $x\neq 0$, 1. The maps x_{xy}, $x<y$ and $]x, y[=\emptyset$ are obtained via isomorphisms of the minimal ideal (which is a circle group) to the group of units (again a circle group) of S_x, for x, $y\neq 0$, 1. If $x=0$ or if $]x, y[\neq\emptyset$, then $m_{xy}=0$, if $y=1$, $m_{xy}(0)=1_x$. Then $\operatorname{Horm}(X, S_x, m_{xy})=S$ may be described geometrically by Figure 4.

Again we have an irreducible semigroup. (This example was discovered by Hunter and Rothman [17] to illustrate that, though the Čech groups of a compact semigroup with zero are trivial, the singular groups need not be. Similar examples were given by them and by Hunter [14] to show, among other things, that in the presence of sufficiently many idempotents, one could not hope always to find nontrivial one-parameter or I-semigroups going to the identity of S.) To see that not all hormoi are irreducible, simply take the product of this semigroup with a compact group.

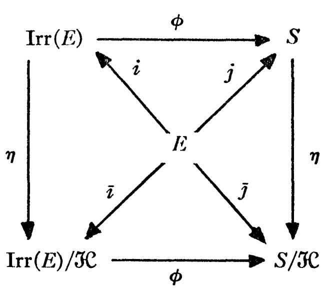
We can now describe the structure of irreducible semigroups as follows:

THEOREM 3. *A semigroup S is an irreducible semigroup if and only if it is isomorphic to a hormos* $\mathrm{Horm}(X, S_x, m_{xy})$ *whose group of units is trivial, and*

$$H_x = (\bigcup \{m_{xy}(S_y): x < y\})^*$$

for each $x \in X$, $x < \max X$. Moreover, in an irreducible semigroup, all maximal subgroups are connected.

Interestingly enough, it is possible to obtain a *universal irreducible* semigroup relative to a given totally ordered set X. This semigroup is denoted by $\mathrm{Irr}(X)$. Thus, if S is an irreducible semigroup, and E denotes the set of idempotents of S, then E is totally ordered relative to the order $e \leqq f$ if $e = ef$, and there is a surmorphism $\bar{\phi}$: $\mathrm{Irr}(E)/\mathcal{H} \to S/\mathcal{H}$ such that the following diagram commutes



where i, j and $\bar{i}$, $\bar{j}$ are all injections.

We can not quite say that $\bar{\phi}$ is an isomorphism, since between two idempotents, there may occur a morphism equivalent to the morphism $\rho(x) = \max(x, 1/2)$.

The Second Fundamental Theorem now takes the following form, where $\mathfrak{D}$ is the equivalence relation on S defined by $(x, y) \in \mathfrak{D}$ iff $I(x) = I(y)$ for $x, y \notin M(S)$, or $x, y \in M(S)$, where $I(x)$ is the principal ideal generated by x.

SECOND FUNDAMENTAL THEOREM. *Let S be a compact semigroup. Then the following are equivalent*:

(1) *The connected component of each idempotent meets the minimal ideal.*

(2) *If* $e^2 = e \notin M(S)$, *then* $H(e)$ *is not open in* eSe.

(3) *If* $e^2 = e \notin M(S)$, *then* $\mathfrak{D}(e)$ *is not open in* SeS.

(4) *If* $e^2 = e$, *then there is a compact connected abelian semigroup in* eSe *containing* e *and meeting the minimal ideal of* S.

(5) *If* $e^2 = e$, *there is an irreducible hormos* T *in* eSe *such that* $T \cap H(e) = e$ *and* $T \cap M(S) \neq \emptyset$.

And so with this theorem, we have our path through the wilderness, and with the previous theorem, we can feel we know every step of the way.

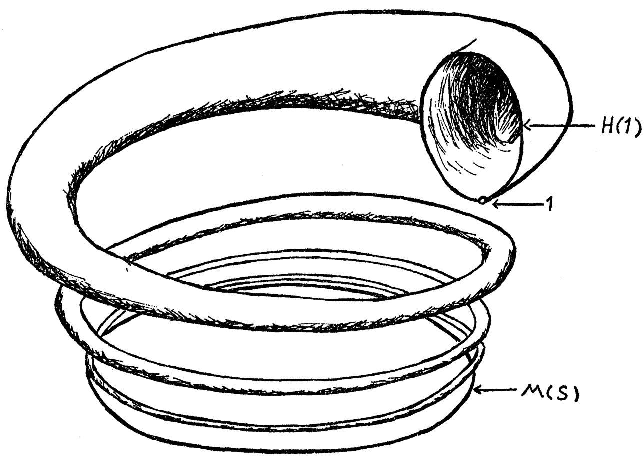


FIGURE 3

Actually, we can say a bit more. We can find T of (5) in the centralizer of any given connected abelian subgroup of $H(e)$. However, it is an unsolved problem to determine if and when T can be found in the centralizer of $H(e)$ itself. We give next some interesting special cases where this can be done.

THEOREM 4. *Let* S *be a compact semigroup with identity and let* G *be a compact subgroup of the group of units. Suppose that the space* S/G *of left cosets* Gs *is a totally ordered connected space, and let* $\pi: S \to S/G$ *be the quotient map. Then*

(a) $sG = Gs$ *for all* $s \in S \backslash M(S)$ *and* π *is a homomorphism of semigroups*;

(b) *there is an I-semigroup* T *containing* 1, *meeting* $M(S)$ *and contained in the centralizer of* G, *and* $\pi | T: T \to S/G$ *is a monomorphism.*

(c) *If* $x \notin M(S)$, *then* $D(x) = Gx$

Further information can easily be obtained in this situation and is given in Hofmann's and my book. Applications to semigroups with $(n-1)$-dimensional group of units embedded in n-manifolds can be made and the results of Mostert and Shields [21] obtained, for example. Results by Hunter [15] and Anderson and Hunter [1] on one (and two)-dimensional semigroups yield with relative ease from these techniques also.

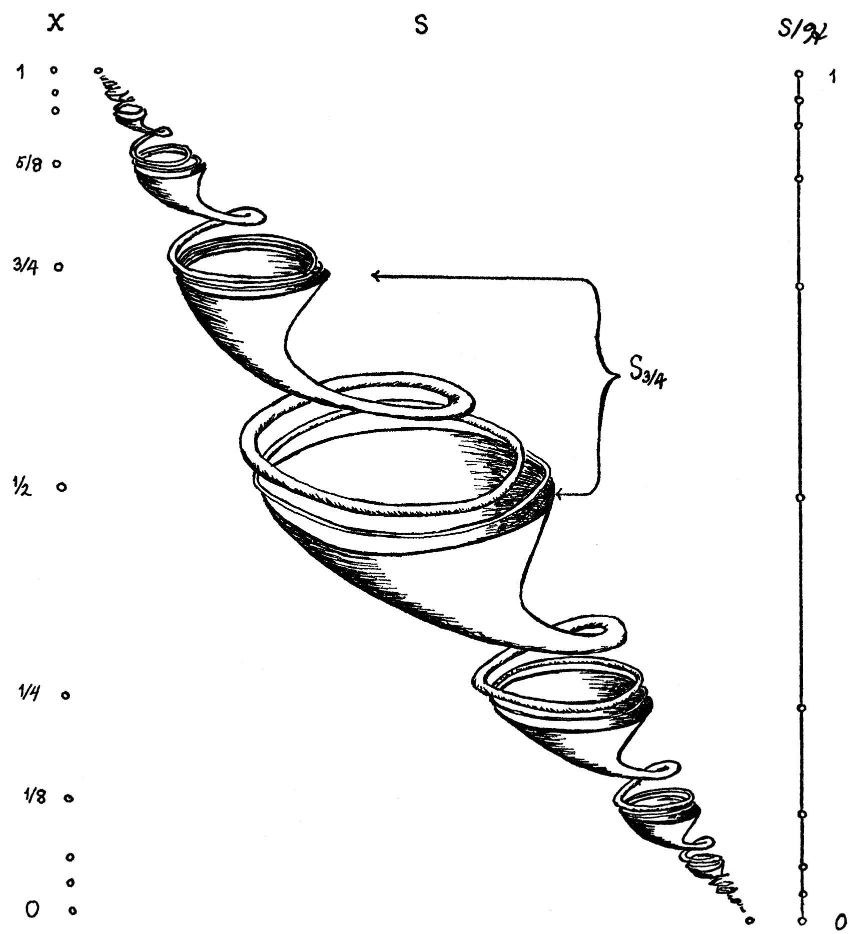


FIGURE 4

Another special case of the centralizing problem is contained in the following result, whose deceivingly simple statement hides formidable consequences—and a correspondingly formidable proof.

THEOREM 5. *Let S be a compact connected semigroup with identity and no other idempotents outside the minimal ideal. Suppose $S/\mathfrak{D}$ is a totally ordered space. Then there is a solenoidal semigroup $T \subset S$ such that*

(a) *T is contained in the centralizer of $H(1)$;*
(b) *$1 \in T$ and $T \cap M(S) \neq \emptyset$.*

This theorem, actually used in the proof of Theorem 4 also, can then be applied to obtain

THEOREM 6. *Let S be a compact semigroup such that $\mathfrak{D}$ is a congruence relation and $S/\mathfrak{D}$ is an I-semigroup. Suppose that for each idempotent $e \notin M(S)$, the set of idempotents in $\mathfrak{D}(e)$ is a finite dimensional space. Then there is an idempotent in the maximal $\mathfrak{D}$-class and, for each such idempotent f, there is an irreducible semigroup T with f as identity such that $T \cap M(S) \neq \emptyset$ and T is in the centralizer of every maximal group $H(e)$ of S with $e^2 = e \in T$. If S is a union of groups, T is an I-semigroup and is unique with respect to the given properties.*

The last statement in the theorem is of some interest in another setting. Let X be a compact connected topological space and T a totally ordered connected space (say the unit interval). Suppose there is a presentation of X as product spaces $X_t \times Y_t = X$, X_t, $Y_t \subset X$, $t \in T$ such that

(i) $X_0 = p = Y_1$,
(ii) if $t < s$, then $X_t \subset X_s$, $Y_s \subset Y_t$,
(iii) $X_t = \bigcap_{s>t} X_s = (\bigcup_{s<t} X_s)^*$,
(iv) $Y_t = (\bigcup_{s>t} Y_s)^* = \bigcap_{s<t} Y_s$.

The finding of an example of a connected compact idempotent semigroup such that $S/\mathfrak{D}$ is totally ordered, but S does not contain unique I-semigroups beginning at each idempotent depends on the finding of an example of such a space. We have shown in our book that for finite dimensional spaces, such a presentation is impossible. John Stallings has supplied us with an example of an infinite dimensional space having such a presentation—namely the space of continuous functions f: $[0, 1] \to [0, 1]$ with $f(0) = 0$ and satisfying the Lipschitz condition $|f(x) - f(y)| \leq |x - y|$.

The question then remains open as to whether or not there is an irreducible semigroup in the centralizer of $H(1)$ if the set of idempotents in $\mathfrak{D}(e)$ is infinite dimensional. The answer, because of Theorem 5, depends entirely on the case when $S/\mathfrak{D}$ is an idempotent semigroup. The question of what happens when $\mathfrak{D}$ is not a congruence but $S/\mathfrak{D}$ is totally ordered also is open. (Some of the results and questions mentioned above were announced without proofs by Hofmann and me [12].)

One of the most important and useful concepts that has been of use in the theory is the notion of *boundary*, or *peripheral points*. In all examples that we know, the group of units lies on a sort of *bounding set*, an adequate definition of which still awaits discovery. A number of notions have been used to describe this situation, but each one seems suited only to special classes of semigroups. In our book Hofmann and I use several of these, although for the main results, we found the following the most useful:

A point p in a compact topological space X is said to be *arc-peripheral* if for any neighborhood U of p, there is a neighborhood V of p, $V^* \subset U$, such that if $z: C \to V \setminus \{p\}$ is a map homotopic in V to a constant map, z is homotopic in $U \setminus \{p\}$ to a constant map. If the homotopy is allowed to be taken over any compact connected space rather than an arc, we say p is $\mathfrak{C}$*-peripheral*, where $\mathfrak{C}$ represents the category of compact connected spaces.

THEOREM 7. *Let S be a compact connected semigroup with identity which is not a group. Then the group of units is $\mathfrak{C}$-peripheral. If there is an arc containing points of the group of units but not contained entirely in the group of units, then the group of units is arc-peripheral.*

The fact that the group of units is arc-peripheral in the circumstances of Theorem 4 was important to the proof of the result. A related result, and one of great importance in the proof of Theorem 4 is the following fact we were forced to prove about groups:

THEOREM 8. *Let G be a compact group. Let $\phi: G \to H$ be an epimorphism, where H is a Lie group, and let B be a closed subgroup of H. Then the natural projection $G \to G/\phi^{-1}(B)$ is homotopic to a constant map if and only if $B = H$.*

It would be valuable to know that this result remains true when the homotopy is taken over an arbitrary connected space instead of an arc.

Let us return now to the concept of peripherality. A good notion of peripherality should minimize the set of peripheral points, but in such a way as to retain Theorem 7. It would be desirable to find a notion which would do this and also imply that every compact connected semigroup of finite dimension has a point which is not peripheral. This would imply the following theorem, due to A. L. Hudson and myself [13]:

THEOREM 9 (HUDSON-MOSTERT). *A compact connected finite dimensional homogeneous semigroup with identity is a group.*

My discussion has been perhaps unduly barren of examples. However, the theory suffers no shortage there. The many fine examples of Hunter and his coworkers have been particularly helpful. Hofmann and I devote a full chapter to examples in our book. Some of them, described for the first time there, were quite valuable to us in our attempts to find the right directions and limits of the theory.

The problems I have mentioned in the above illustrate some of the many interesting and seemingly difficult problems that remain in the theory as we have developed it in our book. Actually, I sometimes feel that the theory may suffer, not from the lack of problems, but from the great plethora of problems that can be asked. Part of the exercise is to find the right ones, perhaps. I tend to feel that, except for the problems mentioned—which are all apparently quite difficult—the study of compact connected semigroups *as a category* has about reached the point of diminishing returns. More restrictive subcategories most likely will present the more interesting questions and satisfying answers. Particular examples would be compact connected abelian semigroups, measure semigroups (for example, the so called Taylor Semigroup [**10**; D-20]), semigroups of endomorphisms of E^n, and semigroups of differentiable or analytic transformations of manifolds. There are very interesting questions in the area of noncompact semigroups, and in semigroups where multiplication is separately, but not jointly, continuous.

BIBLIOGRAPHY

1. L. W. Anderson and R. P. Hunter, *Homomorphisms and dimension*, Math. Ann. **147** (1962), 248–268.

2. A. H. Clifford, *Totally ordered commutative semigroups*, Bull. Amer. Math. Soc. **64** (1958), 305–316.

3. ———, *Naturally totally ordered commutative semigroups*, Amer. J. Math. **76** (1954), 631–646.

4. A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, Vol. I, Math. Surveys, No. 7, Amer. Math. Soc., Providence, 1961.

5. S. Eilenberg, *Sur les groupes compacts d'homéomorphes*, Fund. Math. **28** (1937), 75–80.

6. P. E. Conner, *On the action of the circle group*, Michigan Math. J. **4** (1957), 241–247.

7. B. Gelbaum, G. K. Kalish and J. M. H. Olmstead, *On the embedding of topological semigroups and integral domains*, Proc. Amer. Math. Soc. **2** (1951), 807–821.

8. A. M. Gleason, *Arcs in locally compact groups*, Proc. Nat. Acad. Sci. U.S.A. **36** (1950), 663–667.

9. E. Hewitt and K. A. Ross, *Abstract harmonic analysis*. I, Academic Press, New York, 1963.

10. K. H. Hofmann and P. S. Mostert, *Elements of compact semigroups*, Chas. E. Merrill Co., Columbus, 1966.

11. ———, *Irreducible semigroups*, Bull. Amer. Math. Soc. **70** (1964), 621–627.

12. ———, *Totally ordered* $\mathfrak{D}$*-class decompositions*, Bull. Amer. Math. Soc. **70** (1964), 767–771.

13. A. L. Hudson and P. S. Mostert, *A finite dimensional homogeneous clan is a group*, Ann. of Math. (2) **78** (1963), 41–46.

14. R. P. Hunter, *Note on arcs in semigroups*, Fund. Math. **49** (1961), 233–245.

15. ———, *On one-dimensional semigroups*, Math. Ann. **14** (1962), 383–396.

16. ———, *On the structure of homogroups with applications to the theory of compact connected semigroups*, Fund. Math. **52** (1963), 69–102.

17. R. P. Hunter and N. Rothman, *Characters and cross sections for certain semigroups*, Duke Math. J. **29** (1962), 347–366.

18. K. Iwasawa, *Finite and compact groups*, Sugaku **1** (1948), 30–31. (Japanese)

19. R. J. Koch, *On topological semigroups*, Doctoral Dissertation, Tulane University, New Orleans, La., 1953.

20. ———, *Arcs in partially ordered spaces*, Pacific J. Math. **9** (1959), 723–728.

21. P. S. Mostert and A. L. Shields, *On the structure of semigroups on a compact manifold with boundary*, Ann. of Math. (2) **65** (1957), 117–143.

22. ———, *One-parameter semigroups in a semigroup*, Trans. Amer. Math. Soc. **96** (1960), 510–517.

23. K. Numakura, *On bicompact semigroups with zero*, Bull. Yamagata Univ. (Nat. Sci) **4** (1951), 405–412.

24. ———, *On bicompact semigroups*, Math. J. Okayama Univ. **1** (1952), 99–108.

25. J. E. L. Peck, *An ergodic theorem for a non-commutative semigroup of linear operations*, Proc. Amer. Math. Soc. **2** (1951), 414–421.

26. D. Rees, *On semi-groups*, Proc. Cambridge Philos. Soc. **36** (1940), 387–400.

27. A. D. Wallace, *A note on mobs*. I, An. Acad. Brasil. Ci. **24** (1952), 329–334.

28. ———, *Inverses in euclidean mobs*, Math. J. Okayama Univ. **3** (1953), 23–28.

29. ———, *On the structure of topological semigroups*, Bull. Amer. Math. Soc. **61** (1955), 95–112.

30. ———, *The Rees-Suschkewitch structure theorem for compact simple semigroups*, Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 430–432.