PIECEWISE LINEAR EMBEDDINGS AND ISOTOPIES

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Let M and Q be PL manifolds of dimensions m and q respectively, M being compact. Let ∂M and ∂Q be their boundaries, possibly empty. If $f\colon M\to Q$ is any continuous mapping, $\pi_r(f)$ will denote the relative homotopy group of the pair (C_f, M), C_f being the mapping cylinder of $f\colon M\to Q$.

THE EMBEDDING THEOREM. *Let $f\colon M\to Q$ be a mapping such that $f^{-1}(\partial Q)=\partial M$, and the restriction $f|\partial M$ is a* PL *embedding. If*

$$q - m \geqq 3,$$
$$\pi_r(f) = 0 \quad \textit{for } r \leqq 2m - q + 1,$$
$$\pi_r(M) = 0 \quad \textit{for } r \leqq 3m - 2q + 2,$$

then f is homotopic, keeping ∂M fixed, to a PL *embedding.*

THE UNKNOTTING THEOREM. *Let f, $g\colon M\to Q$ be two* PL *embeddings with $f^{-1}\partial Q=g^{-1}\partial Q=\partial M$ and f and g homotopic keeping ∂M fixed. If*

$$q - m \geqq 3,$$
$$\pi_r(f) = 0 \quad \textit{for } r \leqq 2m - q + 2$$

and

$$\pi_r(M) = 0 \quad \textit{for } r \leqq 3m - 2q + 3,$$

then f and g are PL *ambient isotopic keeping ∂Q fixed.*

These two theorems include the combinatorial analogues of Haefliger's Embedding and Isotopy Theorems of [1] for differential manifolds, as well as Zeeman's Unknotting Theorem [6] and the relevant form of Irwin's Embedding Theorem [5].

The connectivity condition on the map is "best possible," except that some work of Haefliger suggests that homological connectivity may be sufficient. Relaxing the connectivity condition on the map gives rise to an obstruction theory which will be described in a future paper. To what extent the connectivity condition on the manifold is really necessary I have no idea.

The proof of the Embedding Theorem is a lengthy process done by shifting f to general position and then applying a sequence of modifications each of which is fixed outside a PL ball [2].

To prove the Unknotting Theorem, use the homotopy between f and g to construct a map $M \times I \to Q \times I$. Applying the Embedding Theorem gives an embedding $M \times I \to Q \times I$ agreeing with f on $M \times Q$ and $(\partial M) \times I$ and with g on $M \times I$. By Theorem 2 of [3] we can shift this embedding to be level-preserving, and by the Isotopy Extension Theorem f and g are ambient isotopic keeping the boundary fixed.

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