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# $L_{pq}$ INTERPOLATORS AND THE THEOREM OF MARCINKIEWICZ

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The purpose of this paper is to introduce certain interpolation methods (interpolators) which lead to a proof of Marcinkiewicz's theorem. We start with some definitions.

An *interpolation pair* is a couple of Banach spaces continuously contained in a Hausdorff topological vector space $V$.

On the vector spaces $A_1+A_2=\{u\in V: u=v+w,\ v\in A_1,\ w\in A_2\}$ and $A_1\cap A_2$ we introduce the norms

$$\|u\|_{A_1+A_2} = \inf\{\|v\|_{A_1} + \|w\|_{A_2}: v + w = u, v \in A_1, w \in A_2\},$$

$$\|u\|_{A_1\cap A_2} = \max\{\|u\|_{A_1}, \|u\|_{A_2}\};$$

with these norms, $A_1+A_2$ and $A_1\cap A_2$ become Banach spaces.

An *interpolator* $F$ is a function defined on interpolation pairs whose values are Banach spaces $F(A_1, A_2)$ such that:

(1) $A_1\cap A_2\subset F(A_1, A_2)\subset A_1+A_2$, the inclusions being continuous;

(2) if $(X_1, X_2)$, $(Y_1, Y_2)$ are interpolation pairs, and $T$ is a linear map of $X_1+X_2$ into $Y_1+Y_2$ which maps $X_1$ into $Y_1$ and $X_2$ into $Y_2$ and which decreases the norms, then $T$ is also a norm decreasing map of $F(X_1, X_2)$ into $F(Y_1, Y_2)$.

We will say that $F(A_1, A_2)$ is an intermediate space between $A_1$ and $A_2$.

The functions considered in the following are complex-valued functions defined on a totally $\sigma$-finite measure space $(M, m)$. The distribution function of $f$ is

$$D(f, \lambda) = m(\{x \in M: |f(x)| > \lambda\});$$

the nonincreasing rearrangement of $f$ is defined by

$$f^*(t) = \inf\{\lambda: D(f, \lambda) \leqq t\},$$

and the average function of $f$ by

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\, ds.$$

We define

$$(1)\ \|f\|_{L_{pq}} = \begin{cases} \left\{\dfrac{q}{pp'}\displaystyle\int_0^\infty f^{**}(t)^q t^{q/p-1}\, dt\right\}^{1/q} & \text{if } 1 < p < \infty,\ 1 \leqq q < \infty, \\ \sup\limits_{t>0} t^{1/p} f^{**}(t) & \text{if } 1 \leqq p < \infty,\ q = \infty, \\ \sup\limits_{t>0} t f^{**}(t) = \displaystyle\int_0^\infty f^*(t)\, dt & \text{if } p = 1,\ 1 \leqq q \leqq \infty, \\ \sup\limits_{t>0} f^{**}(t) & \text{if } p = \infty,\ 1 \leqq q \leqq \infty. \end{cases}$$

For $1 < p \leqq \infty$, the set $\{f \in L^1 + L^\infty: \|f\|_{L_{pq}} < \infty\}$ is the well-known Lorentz space $L_{pq}$ (see, for instance, [3]) with a slightly different norm. With our definition, the $L_{pq}$ norm of a function $f \in L^1 \cap L^\infty$ is a continuous function of $(p, q)$ for $1 \leqq p \leqq \infty$, $1 \leqq q \leqq \infty$, and reduces to the $L^1$ norm for $p = 1$. A direct computation shows that $\|f\|_{L_{pq}} = \|f\|_{L^p}$ if $f$ is the characteristic function of a measurable subset of $M$. Another advantage of our definition is that the $L_{p1}$ and $L_{p\infty}$ spaces can be characterized as the minimum and maximum among the intermediate spaces $C$ between $L^1$ and $L^\infty$ such that $\|f\|_C = \|f\|_{L^p}$ for every characteristic function $f$. More precisely:

THEOREM 1. *Let $1 \leqq p < \infty$. Let $C$ be an intermediate space between $L^1$ and $L^\infty$ such that $\|f\|_C = \|f\|_{L^p}$ for every characteristic function $f$. Then $L_{p1} \subset C \subset L_{p\infty}$ continuously, and*

$$\|f\|_{L_{p\infty}} \leqq \|f\|_C \leqq \|f\|_{L_{p1}},$$

*the first inequality being valid for every $f \in C$ and the second for every $f \in L_{p1}$.*

If $t$ is a positive real number, we denote by $tA$ the Banach space obtained from $A$ by changing its norm $\|\ \|_A$ by

$$\|\ \|_{tA} = 1/t\|\ \|_A.$$

The following theorem has been proved in [2].

THEOREM 2. *If* $f\in L^1+L^\infty$ *and* $t>0$, *then* $f^{**}(t)=\|f\|_{tL^1+L^\infty}$.

This result leads to a natural generalization of the average function.

DEFINITION. If $(A_1, A_2)$ is an interpolation pair and $u\in A_1+A_2$, we define the average function of $u$ with respect to $(A_1, A_2)$ as the function

$$u^{**}(t) = u^{**}(A_1, A_2; t) = \|u\|_{tA_1+A_2}.$$

We now list some properties of $u^{**}(t)$.

(1) For every $t>0$, $u^{**}(t)$ is finite.

(2) $u^{**}(t)$ is nonincreasing and continuous, $tu^{**}(t)$ is nondecreasing.

(3) $u^{**}(A_1, A_2; t)=(1/t)u^{**}(A_2, A_1; 1/t)$.

(4) $u^{**}(t)$ tends to zero when $t$ tends to infinity for every $u$ in $A_1+A_2$ if and only if $A_1\cap A_2$ is dense in $A_2$.

(5) $tu^{**}(t)$ tends to zero when $t$ tends to zero for every $u$ in $A_1+A_2$ if and only if $A_1\cap A_2$ is dense in $A_1$.

(6) $\sup_{t>0} u^{**}(t)\leqq\|u\|_{A_2}$, the equality sign holds for every $u$ in $A_2$ if and only if the unit sphere of $A_2$ is closed in $A_1+A_2$.

(7) $\sup_{t>0} tu^{**}(t)\leqq\|u\|_{A_1}$, and the equality sign holds for every $u$ in $A_1$ if and only if the unit sphere of $A_1$ is closed in $A_1+A_2$.

(8) Let $u\in A_1+A_2$ and denote

$$\phi(\delta) = \inf\{\|w\|_{A_2}: (1/t)\|u - w\|_{A_1} + \|w\|_{A_2} \leqq u^{**}(t) + \delta\},$$
$$u^{*}(t) = u^{*}(A_1, A_2; t) = \lim_{\delta\to 0} \phi(\delta).$$

Then $u^*(t)$ is nonincreasing and right continuous, and

$$u^{**}(A_1, A_2; t) = \frac{1}{t}\int_0^t u^{*}(A_1, A_2; s)\, ds.$$

We define, for $u\in A_1,+A_2$ $\|u\|_{L_{pq}(A1,\, A2)}$ as in (1), but writing $u^{**}(A_1, A_2; t)$ instead of $f^{**}(t)$. It can be shown that $\|u\|_{L_{pq}(A_1,A_2)}$ is a norm, and denoting

$$L_{pq}(A_1, A_2) = \{u \in A_1 + A_2: \|u\|_{L_{pq}(A_1,A_2)} < \infty\},$$

the function $(A_1, A_2)\to L_{pq}(A_1, A_2)$ turns out to be an interpolator.

We now compute the average function in some particular cases.

THEOREM 3. *Let* $1\leqq p_1<p_2<\infty$, $1/\beta=1/p_1-1/p_2$; *then*

$$f^{**}(L_{p_1,1}, L_{p_2,1}; t)$$
$$= \frac{1}{p_1 t}\int_0^{t^\beta} f^*(s)s^{1/p_1-1}\, ds + \frac{1}{p_2}\int_{t^\beta}^\infty f^*(s)s^{1/p_2-1}\, ds. \tag{2}$$

*For* $p_2 = \infty$, *the right-hand side of* (2) *reduces to the first term, with* $\beta = p_1$.

THEOREM 4. *Let* $1 \leqq q_1,\ q_2 \leqq \infty,\ q_1 \neq q_2$; *then*

$$H(f) \leqq f^{**}(L_{q_1,\infty}, L_{q_2,\infty}; t) \leqq 2H(f),$$

*where*

$$H(f) = \inf_{0 \leqq \alpha \leqq 1} \sup_{s>0} \frac{f^{**}(s)}{(1-\alpha)ts^{-1/q_1} + \alpha s^{-1/q_2}} \cdot$$

Using these results it is not difficult to determine the result of applying an $L_{pq}$ interpolator to couples of Lorentz spaces.

THEOREM 5. *Let* $1 \leqq p_1 < p_2 \leqq \infty$, $1 < k < \infty$, $1 \leqq r \leqq \infty$,

$$(3) \qquad 1/p = 1/(p_1 k) + 1/(p_2 k'), \qquad 1/\beta = 1/p_1 - 1/p_2.$$

*Then* $L_{pr}$ *is continuously contained in* $L_{kr}(L_{p_1,1}, L_{p_2,1})$, *and*

$$\|f\|_{L_{kr}(L_{p_1,1}, L_{p_2,1})} \leqq \left(\frac{\beta k k'}{p p'}\right)^{1/r'} p' \|f\|_{L_{pr}}.$$

THEOREM 6. *Let* $1 \leqq q_1,\ q_2 \leqq \infty,\ q_1 \neq q_2,\ M_1,\ M_2 > 0,\ 1 < k < \infty,$ $1 \leqq r \leqq \infty$, *and*

$$(4) \qquad 1/q = 1/(q_1 k) + 1/(q_2 k'), \qquad 1/\gamma = 1/q_1 - 1/q_2.$$

*Then* $L_{kr}(M_1 L_{q_1\infty}, M_2 L_{q_2\infty})$ *is continuously contained in* $L_{qr}$, *and*

$$\|f\|_{L_{qr}} \leqq \left(\frac{|\gamma| k k'}{q q'}\right)^{1/r} M_1^{1/k} M_2^{1/k'} \|f\|_{L_{kr}(M_1 L_{q_1\infty}, M_2 L_{q_2\infty})}.$$

The following theorem is an immediate consequence of Theorems 5 and 6.

THEOREM 7. *Let* $T$ *be a linear operator defined on* $L^1 \cap L^\infty$ *such that* $\|Tf\|_{L_{q_i\infty}} \leqq M_i \|f\|_{L_{p_i 1}}$ *for every* $f \in L^1 \cap L^\infty$, $i = 1, 2$, *with* $1 \leqq p_i \leqq \infty$, $1 \leqq q_i \leqq \infty$, $p_1 < p_2$, $q_1 \neq q_2$. *Then, if* $1 \leqq r \leqq \infty$, $1 < k < \infty$, $T$ *can be extended to a continuous linear operator from* $L_{pr}$ *into* $L_{qr}$ *such that*

$$(5) \qquad \|Tf\|_{L_{qr}} \leqq M_1^{1/k} M_2^{1/k'} k k' \frac{|\gamma|^{1/r} \beta^{1/r'}}{(qq')^{1/r} (pp')^{1/r'}} p' \|f\|_{L_{pr}}$$

*where* $\beta$ *and* $\gamma$ *are the quantities defined in* (3) *and* (4).

Theorem 7, due to A. P. Calderón (1), implies Marcinkiewicz's theorem except for the case $p_1 = q_1 = 1$. In order to include this case we would need an inequality similar to (5), but replacing $\|Tf\|_{L_{qr}}$ in the left-hand side by

$$\left\{\int_0^\infty (Tf)^*(t)^q t^{q/p-1}\,dt\right\}^{1/q}.$$

Such an inequality has been obtained, even with more generality (see [4]). We can also obtain it without essential modification of our methods using the fact that

$$(g+h)^*(t) \leqq g^*(t/2) + h^*(t/2). \tag{6}$$

However, the interest of our proof lies in the fact that Theorem 7 has been obtained from the general theory of interpolation and we don't yet know if an inequality similar to (6) holds in general for $u^*(A_1, A_2; t)$.

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