A NOTE ON FREE SUBSEMIGROUPS WITH TWO GENERATORS[1]

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Let Σ be an alphabet and $F(\Sigma)$ the free semigroup of words over Σ [3]. Let $E=\{w_1, w_2\}\subset F(\Sigma)$, where w_1 and w_2 are nonempty words. We give a necessary and sufficient condition on E in order that the subsemigroup generated by E be free with E as its unique irreducible generating set.

DEFINITION. If $w, z\in F(\Sigma)$ are such that $w=z^n$, $n\geqq 0$, then z is a *root* of w. The root of w having minimum word length (maximum n) is the *primitive root* of w and is denoted by $\rho(w)$.

REMARK. It follows from results in [2], [3], [4] that $\rho(w^k)=\rho(w)$ for any $k>0$. Also, if $w_1w_2=w_2w_1$ and w_1, w_2 are both nonempty words, then $\rho(w_1)=\rho(w_2)$. The converse is obviously true; i.e. if $\rho(w_1)=\rho(w_2)$, then $w_1w_2=w_2w_1$.

THEOREM. *Let $E=\{w_1, w_2\}$ be a set of two nonempty words in $F(\Sigma)$. The subsemigroup generated by E is free with E as its unique irreducible generating set if and only if $\rho(w_1)\neq\rho(w_2)$.*

PROOF. Suppose $\rho(w_1)=\rho(w_2)=z$; then by the remark above, $w_1w_2=w_2w_1$ is a nontrivial relation which implies that the subsemigroup generated by E is not a free semigroup with E as its unique irreducible generating set.

Now suppose that the subsemigroup generated by E is not free with E as its unique irreducible generating set. Then there exists a nontrivial relation of the form

$$w_1^{n_1}w_2^{n_2}\cdots = w_2^{m_1}w_1^{m_2}\cdots, \tag{1}$$

with $n_i>0$, $m_i>0$. Without loss of generality, we may assume that the length of w_2 is not greater than the length of w_1. This together with (1) implies that w_2 is a left-factor of w_1. Hence, we have

$$w_1 = w_2^q t_0, \tag{2}$$

where $q\geqq 1$ is taken to be such that w_2^q is the largest power of w_2 which is a left-factor of w_1. Substituting in (1), we obtain $(w_2^qt_0)w_2\cdots = w_2^{m_1}(w_2^qt_0)\cdots$,

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$$(3) \qquad t_0 w_2 \cdots = w_2^{m_1} t_0 \cdots .$$

By the definition of q, w_2 cannot be a left-factor of t_0. Therefore, by (3), t_0 must be a left-factor of w_2. We may write $w_2 = t_0 t_1$.

Equation (1) clearly has the form $vw_1 = v'w_2$, where v, $v' \in F(\Sigma)$. This implies that w_2 is a right-factor of w_1. However, $w_1 = w_2^{q-1} t_0 t_1 t_0$. Since the length of $t_1 t_0$ is equal to the length of w_2, it follows that $w_2 = t_1 t_0$. Thus, $t_1 t_0 = t_0 t_1$ and by the remark, $\rho(t_0) = \rho(t_1) = z$. This implies $\rho(w_2) = z$ and by (2) we also have $\rho(w_1) = z$. This completes the proof.

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CONDITIONAL INTEGRABILITY OVER MEASURE SPACES

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Introduction. Let E^n be the n-dimensional Euclidean space, $(A, \mathcal{A}, \mu)$ be a measure space and f be a measurable function from A into E^n. According to the theory of integration the function f is either integrable or not integrable. The aim of this note is to show that nonintegrable functions can be divided into two classes: the totally unintegrable functions and the conditionally integrable functions. Moreover we shall show that the conditionally integrable functions may be further characterized by an index of conditional integration which can take the values $1, 2, \cdots, n$.

Notations. A subset $\mathfrak{N}$ of $\mathcal{A}$ is a nest if for any N_1 and $N_2 \in \mathfrak{N}$ we have either $N_1 \subset N_2$ or $N_2 \subset N_1$. A nest $\mathfrak{N}$ of $\mathcal{A}$ is called a sweeping nest for the measure space $(A, \mathcal{A}, \mu)$ if for each set $D \in \mathcal{A}$ with $\mu(D) < \infty$ and for each $\epsilon > 0$ there exists a set $N \in \mathfrak{N}$ such that