A COMBINATORIAL THEOREM FOR STOCHASTIC PROCESSES

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Let $\{\chi(u), 0 \leqq u \leqq t\}$ be a stochastic process where t is a finite positive number. We associate a stochastic process $\{\chi^*(u), 0 \leqq u < \infty\}$ with $\{\chi(u), 0 \leqq u \leqq t\}$ as follows: $\chi^*(u) = \chi(u)$ for $0 \leqq u \leqq t$ and $\chi^*(t+u) = \chi^*(t) + \chi^*(u)$ for $u > 0$. If the finite dimensional distributions of $\{\chi^*(v+u) - \chi^*(v), 0 \leqq u \leqq t\}$ are independent of v for $v \geqq 0$, then the process $\{\chi(u), 0 \leqq u \leqq t\}$ is said to have *cyclically interchangeable increments*. In particular, if $\{\chi(u), 0 \leqq u \leqq t\}$ has stationary, independent increments, and $P\{\chi(0) = 0\} = 1$, then it belongs to this class.

THEOREM. *If $\{\chi(u), 0 \leqq u \leqq t\}$ is a separable stochastic process with cyclically interchangeable increments and if almost all sample functions are nondecreasing step functions which vanish at $u = 0$, then*

$$P\{\chi(u) \leqq u \text{ for } 0 \leqq u \leqq t \mid \chi(t)\} = \begin{cases} \left(1 - \dfrac{\chi(t)}{t}\right) & \text{if } 0 \leqq \chi(t) \leqq t, \\ 0 & \text{otherwise}, \end{cases} \tag{1}$$

with probability 1.

PROOF. Let $\chi^*(u)$, $0 \leqq u < \infty$, be a nondecreasing step function (nonrandom) for which $\chi^*(0) = 0$ and $\chi^*(t+u) = \chi^*(t) + \chi^*(u)$ if $u > 0$ where t is a fixed positive number. For $u \geqq 0$ define

$$\xi(u) = \begin{cases} 1 & \text{if } \chi^*(v) - \chi^*(u) \leqq v - u \text{ for } v \geqq u, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Obviously $\xi(u+t) = \xi(u)$ for all $u \geqq 0$. Now we shall prove that

$$\int_0^t \xi(u)\,du = \begin{cases} t - \chi^*(t) & \text{if } 0 \leqq \chi^*(t) \leqq t, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

The case $\chi^*(t) \geqq t$ is obvious. Thus we suppose that $0 \leqq \chi^*(t) < t$. Define

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(4) $$\alpha(c) = \sup\{u: u - \chi^*(u) < c \text{ and } 0 \leqq u < \infty\}$$

for $c>0$. The function $\alpha(c)$ is increasing. $\alpha(c)$ increases either linearly with slope 1 or by jumps. Evidently $\alpha(c-0)=\alpha(c)$ and $\alpha(c+t-\chi^*(t)) = \alpha(c)+t$ for $c>0$. If $u=\alpha(c)$, then $\xi(u)=1$. Since $\alpha(c)-\chi^*(\alpha(c))=c$ and $u-\chi^*(u)\geqq c$ for $u\geqq\alpha(c)$, it follows that $\chi^*(u)-\chi^*(\alpha(c))\leqq u-\alpha(c)$ for $u\geqq\alpha(c)$, that is, $\xi(\alpha(c))=1$. If $\alpha(c)<u<\alpha(c+0)$, then evidently $\xi(u)=0$. Accordingly, in the interval $[\alpha(c), \alpha(c+t-\chi^*(t))]$ of length t, $\xi(u)=1$ on the set $\{u: u=\alpha(z) \text{ for } c\leqq z\leqq c+t-\chi^*(t)\}$ of measure $t-\chi^*(t)$ and $\xi(u)=0$ elsewhere. (If $\chi^*(u)$ is defined as continuous on the right, and z is a discontinuity point of $\alpha(z)$, then also $\xi(\alpha(z+0)) =1$. However, since the discontinuity points of $\alpha(z)$ form a set of measure 0, this does not make any difference.) Since $\xi(u)$ is periodic with period t, (3) follows.

Now, if we suppose that $\{\chi^*(u), 0\leqq u<\infty\}$ is the stochastic process associated with the process $\{\chi(u), 0\leqq u\leqq t\}$ under consideration and if $\xi(u)$ is defined by (2), then $\xi(u)$ is a random variable which has the same distribution for all $u\geqq 0$. Thus

$$\begin{aligned}
&P\{\chi(u) \leqq u \text{ for } 0 \leqq u \leqq t \mid \chi(t)\} \\
&\quad = E\{\xi(0) \mid \chi(t)\} = \frac{1}{t}\int_0^t E\{\xi(u) \mid \chi(t)\}\,du \\
&\quad = E\left\{\frac{1}{t}\int_0^t \xi(u)\,du \mid \chi(t)\right\} = \begin{cases} \left(1 - \dfrac{\chi(t)}{t}\right) & \text{if } 0 \leqq \chi(t) \leqq t, \\ 0 & \text{otherwise,} \end{cases}
\end{aligned}$$

with probability 1. The last equality follows from identity (3) which now holds for almost all sample functions. This completes the proof of the theorem.

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