RESEARCH PROBLEMS

4. R. A. Hirschfeld: *Invariant subspaces.*

E is a complex locally convex vector space, in which every closed bounded subset is complete. Let $T: E \to E$ be a linear continuous operator with nonempty spectrum, possessing a continuous inverse $T^{-1}: E \to E$.

Assume the family $(T^n)_{n=-\infty}^{\infty}$ to be equicontinuous.

Is it true that there is an invariant closed nontrivial linear subspace for T? (For a Banach space the answer is yes.) (Received December 4, 1964.)

5. R. A. Hirschfeld: *Extension of nonlinear contractions.*

E and F are Banach spaces, F reflexive, D is a subset of E and $T: D \to F$ a nonlinear contraction, i.e., $\|Tx_1 - Tx_2\|_F \leqq \|x_1 - x_2\|_E$ whenever $x_1, x_2 \in D$.

Can T be extended to a contraction $\tilde{T}: E \to F$? (For $E = F =$ Hilbert space the answer is yes.) (Received December 4, 1964.)

6. Richard Bellman: *Factorization of linear differential operators modulo p.*

Let D represent the operator d/dx. Consider the factorization

$$D^2 + a_1(x)D + a_2(x) = (D + b_1(x))(D + b_2(x)),$$

where a_1, a_2, b_1, and b_2 are polynomials in x of degree less than p, a prime, and the equality is required to hold modulo p. What is the number of irreducible linear differential operators for the case where $a_1(x)$ and $a_2(x)$ are required, respectively, to have degrees m_1 and m_2? Generalize to the case of linear differential operators of the form $D^n + a_1(x)D^{n-1} + \cdots + a_n(x)$. (Received November 30, 1964.)

7. Richard Bellman: *Functional differential equations.*

Under what condition on the function $r(t) \geqq 0$ can one assert that all solutions of $u'(t) + au(t - r(t)) = 0$ approach zero as $t \to \infty$?

Under what conditions do all solutions of $u'(t) = au(t - r(t)) = \sin bt$ approach $c \sin bt$ as $t \to \infty$?

If all solutions of $u'(t) + au(t - r) = 0$ approach zero as $t \to \infty$, and if $|r(t) - r| \leqq \epsilon$ for $t \geqq 0$, do all solutions of $u'(t) + au(t - r(t)) \to 0$, as $t \to \infty$, for ϵ sufficiently small? (Received November 30, 1964.)

8. Richard Bellman: *Lagrange expansion for operators.*

It has been recognized in recent years [cf. Good, *Generalizations to several variables of Lagrange's expansion, with applications to stochastic processes*, Proc. Cambridge Philos. Soc. 56 (1960), 367–380] that the multidimensional Lagrange expansion is a very useful tool in many parts of analysis and mathematical physics.

Regarding the nonlinear integral equation,

$$u(x) = f(x) + t\int_0^1 k(x, y)\phi(u(y))\,dy, \tag{1}$$

as a limiting form of the simultaneous system of equations,

$$u(x_i) = f(x_i) + t\sum_{j=1}^{N} w_j k(x_i, x_j)\phi(u(x_j)), \tag{2}$$

obtained by numerical quadrature, we can obtain an expansion of the form

$$\psi(u) = \psi(f) + t\psi_1 + \cdots + t^n\psi_n + \cdots, \tag{3}$$

which is analogous to the Lagrange expansion. Can one obtain this expansion in a simpler fashion?

Generally, considering a nonlinear equation

$$u = f + tN(u), \tag{4}$$

where N is a nonlinear operation, can one obtain a generalized Lagrange expansion in terms of functional derivatives? (Received December 4, 1964.)

9. Richard Bellman: *Functional equations and dynamic programming.*

Let $q_1(x)$ and $q_2(x)$ be quadratic polynomials which approach $+\infty$ and $x\rightarrow+\infty$. An important role in dynamic programming is played by the easily established fact that

$$\min_y[q_1(x - y) + q_2(y)] = q_3(x),$$

where $q_3(x)$ is again a quadratic polynomial in x. Are these the only functions which display this invariance under the foregoing composition rule? Specifically,

(a) Let $q(x, a)$ be a function of a scalar x and an N-dimensional vector a. To what extent are q and ϕ determined by the relation

$$\min_y[q(x - y, a) + q(y, b)] = q(x, \phi(a, b)),$$

where ϕ is an N-dimensional function of a and b?

(b) What functions allow the composition

$$\min_{y}[q(h_1(x, y), a) + q(h_2(x, y), b)] = q(x, \phi(a, b)),$$

under the further condition that the minimizing value is uniquely determined?

(c) What are the appropriate multidimensional versions of these problems—and solutions?

For a background discussion and some results, one can refer to Chapter X of R. Bellman and S. Dreyfus, *Applied dynamic programming*, Princeton Univ. Press, Princeton, N. J., 1962. (Received December 7, 1964.)