HOLOMORPHIC CONVEXITY OF TEICHMÜLLER SPACES[1]

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Let B be the complex Banach space of holomorphic functions $\phi(z)=\phi(x+iy)$ defined for $y<0$, with norm $\|\phi\|=\sup|y^2\phi(z)|$. The universal Teichmüller space T may be considered as a subset of B defined as follows [2], [7]. $\phi\in B$ belongs to T if and only if there is a quasiconformal selfmapping $w(z)$ of the z-plane which leaves 0 and 1 fixed and is, for $y<0$, a conformal mapping with Schwarzian derivative $\phi(z)$. If this is the case we say that w belongs to ϕ. T is a bounded domain in B containing the origin. The so-called Teichmüller metric (see below) is defined in T; it is topologically equivalent to the metric of B. Every boundary point of T has infinite Teichmüller distance from the origin.

If $Q\subset T$, we denote by $h(Q)$ the hull of Q with respect to continuous holomorphic functions in T. $\psi\in T$ belongs to $h(Q)$ if and only if there is no continuous holomorphic function f in T such that $|f(\psi)|>|f(\phi)|$ for all $\phi\in Q$.

THEOREM 1. *If $Q\subset T$ is bounded in the Teichmüller metric, so is $h(Q)$.*

PROOF. For $\phi\in T$ let $K(\phi)$ denote the smallest dilitation of a mapping w belonging to ϕ. The function $K(\phi)$ is well defined and $\log K(\phi)$ is the Teichmüller distance of ϕ to the origin.

For $\phi\in T$ and any three real numbers $a<b<c$ set $f_{a,b,c}(\phi)=(w(b)-w(a))/(w(c)-w(a))$ where w is any mapping belonging to ϕ. These functions are well defined and one verifies, using [3], that they are continuous and holomorphic in T.

Let $\phi\in T$ and $K(\phi)\leqq\alpha$. Then there is a w belonging to ϕ with dilitation not exceeding α. Let Γ be the image of the real axis under w; this curve depends only on ϕ. Set $\chi(\zeta)=w(w^{-1}(\zeta)^*)$ where the asterisk denotes complex conjugation. Then χ is a quasireflection about Γ, that is an orientation-reversing topological selfmapping of the plane which leaves every point of Γ fixed, and the dilitation of χ is at most α^2. By a theorem of Ahlfors [2] it follows that $|f_{a,b,c}(\phi)|\leqq\beta$ for all $a<b<c$, where β depends only on α.

Assume now that $|f_{a,b,c}(\phi)|\leqq\alpha$ for all $a<b<c$ and let Γ be the image of the real axis under a mapping w belonging to ϕ. Again by

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[2], there exists a quasireflection χ about Γ of dilitation not exceeding β, where β depends only on α. Set $w_1(x+iy)=w(x+iy)$ for $y\leqq 0$ and $w_1(x+iy)=\chi(w(x-iy))$ for $y>0$. Then w_1 belongs to ϕ and the dilitation of w_1 is at most β.

We conclude that if $K\leqq\alpha<\infty$ on a set $Q\subset T$, then $K\leqq\alpha'<\infty$ on $h(Q)$ where α' depends only on α. This proves the theorem.

Let G be a Fuchsian group with the real axis as fixed line. Let $B(G)$ be the closed subspace of B consisting of those elements for which $\phi(z)dz^2$ is invariant under G. The Teichmüller space $T(G)$ of quasiconformal deformations of G may be identified with a domain in $B(G)$, namely the component of $T\cap B(G)$ containing the origin [6], [7], [8]. We note that dim $B(G)<\infty$ if and only if G is finitely generated and of the first kind.

If $Q\subset T(G)$ is bounded in the Teichmüller metric, the same is true of the hull $h_G(Q)$ of Q with respect to continuous holomorphic functions in $T(G)$, for $h_G(Q)\subset h(Q)$. If dim $B(G)<\infty$, this means that $T(G)$ is holomorphically convex in the sense of Cartan-Thullen; thus we obtain

THEOREM 2. *Finite-dimensional Teichmüller spaces are domains of holomorphy.*

This theorem applies, in particular, if G is a hyperbolic group representing a closed Riemann surface of genus $g>1$. In this case $T(G)$ may be identified with the Teichmüller space T_g of such Riemann surfaces, and Theorem 2 yields some information on period matrices.

Let H_g denote the set of symmetric $(g\times g)$ matrices with positive definite imaginary parts (Siegel's generalized upper half-plane). There exists a natural holomorphic mapping $t\rightarrow Z(t)$ of T_g into H_g defined as follows. A point $t\in T_g$ may be considered as (the conformal equivalence class of) a closed Riemann surface $S(t)$ together with a "standard" system of generators $\{a_1, b_1, a_2, \cdots, b_g\}$ of the fundamental group of $S(t)$. This system of generators defines a canonical homology basis $\{\alpha_1, \beta_1, \cdots, \beta_g\}$ on $S(t)$. We set $Z(t)=(Z_{ij})$, where Z_{ij} is the period over β_j of the Abelian differential of the first kind on $S(t)$ which has periods 1 over α_i and 0 over α_k, $k\neq i$.

We denote the closure of $Z(T_g)$ in H_g by A_g and set $B_g=A_g-Z(T_g)$. By Baily's theorem [4], A_g and B_g are analytic sets.

THEOREM 3. *For $g>3$, every point $b\in B_g$ with* $\dim_b B_g<3g-4$ *is a singular point of A_g.*

PROOF. It is known [1], [5], [9] that dim $T_g=3g-3$ and that the points $t\in T_g$ such that $S(t)$ is hyperelliptic form a $(2g-1)$-dimensional subvariety T_g'. Also, the rank of Z is $3g-3$ at all points of T_g-T_g' and $2g-1$ at points of T_g' (so that $A_2=H_2$, $A_3=H_3$). We denote by C_g the closure of $Z(T_g')$ in H_g.

Let $g>3$ and assume that there is a $b\in B_g$ with $\dim_b B_g=q\leqq 3g-5$ which is a regular (simple) point of A_g. Then $b\notin C_g$ and there exists a neighborhood M of b in H_g such that M does not meet C_g, $M\cap A_g$ is holomorphically homeomorphic to a simply-connected domain in a $(3g-3)$-dimensional number space, and $\dim(M\cap B_g)=q$. Set $N=(M\cap A_g)-(M\cap B_g)$. Then N is a simply-connected domain in $Z(T_g)$ and there exists a domain $N_0\subset T_g$ such that $Z|N_0$ is a homeomorphism onto N. There is a sequence $\{t_j\}\subset N_0$ such that $\lim Z(t_j)$ is b. This sequence diverges in T_g so, by Theorem 2, there is a holomorphic function f in T_g such that the sequence $\{f(t_j)\}$ is unbounded. But $f|N_0=F\circ(Z|N_0)$, where F is holomorphic in N and hence, by Hartogs' theorem, can be continued analytically over $M\cap A_g$. Thus the sequence $\{F(Z(t_j))\}$ converges. Contradiction.

REMARK. It is very likely that dim $B_g=3g-5$ (though there seems to be no proof of this in the literature). If so, our result can be restated as

THEOREM 3′. *For $g>3$ the singular set of A_g is $B_g\cup C_g$.*

Note added in proof (*September* 20, 1964). Theorem 3′ is indeed valid since dim $B_g=3g-5$. More precisely, every element of B_g is of the form $\beta(Z)$ where Z is a direct sum of $k>1$ matrices $Z_j\in A_{g_j}$ with $g_1+\cdots+g_k=g$ and β is an element of the group Γ_g of holomorphic automorphisms of H_g defined by integral symplectic matrices. A derivation of this statement from the results of Matsusaka and Hoyt (see [4] for references) has been communicated to us by Mumford. The key fact is that for a fixed integer $n>3$ the complex manifold $X=H_g/\Gamma_g(n)$, where $\Gamma_g(n)$ denotes the subgroup of Γ_g corresponding to matrices congruent to the identity mod n, can be realized as a Zariski open subset of a projective algebraic variety, and that the canonical family of polarized Abelian varieties defined over H_g is induced by an algebraic family defined over X.

We also note that the proof of Theorem 3 can be refined so as to give the following sharper result. Let b be a point of B_g but not of C_g, and let M be a connected neighborhood of b in H_g which does not meet C_g. Then the intersection of M with A_g-B_g has an infinite fundamental group.

REFERENCES

1. Lars Ahlfors, *The complex analytic structure of the space of closed Riemann surfaces*, pp. 45–66, Analytic functions, Princeton Univ. Press, Princeton, N. J., 1960.

2. ———, *Quasiconformal reflections*, Acta Math. **109** (1963), 291–301.

3. Lars Ahlfors and Lipman Bers, *Riemann's mapping theorem for variable metrics*, Ann. of Math. (2) **72** (1960), 385–404.

4. W. L. Baily, Jr., *On the moduli of Jacobian varieties*, Ann. of Math. (2) **71** (1960), 303–314.

5. Lipman Bers, *Holomorphic differentials as functions of moduli*, Bull. Amer. Math. Soc. **67** (1961), 206–210.

6. ———, *Correction to "Spaces of Riemann surfaces as bounded domains,"* Bull. Amer. Math. Soc. **67** (1961), 465–466.

7. ———, *Automorphic forms and general Teichmüller spaces*, Proc. Conf. Compl. Anal., Univ. of Minnesota, Minneapolis, Minn., Univ. of Minnesota Press, (to appear).

8. C. J. Earle, *The Teichmüller space of an arbitrary Fuchsian group*, Bull. Amer. Math. Soc. **70** (1964), 699–701.

9. H. E. Rauch, *On the transcendental moduli of algebraic Riemann surfaces*, Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 42–49.

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