IN-GROUPS AND IMBEDDINGS OF n-COMPLEXES IN $(n+1)$-MANIFOLDS

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Let K^n denote an n-dimensional subcomplex of a closed orientable $(n+1)$-manifold, M^{n+1}.

Denote the n-simplices of K^n by $\tau_1, \tau_2, \cdots, \tau_p$, and the $(n-1)$-simplices of K^{n-1} by $\sigma_1, \sigma_2, \cdots, \sigma_q$. Let F denote the *free* (not free abelian) group generated by $\tau_1, \tau_2, \cdots, \tau_p$. Assume M^{n+1}, the τ_i and σ_j have been oriented. Let l_j be a nice small loop about σ_j, oriented in such a way that the orientation of l_j and σ_j taken together agrees with that of M^{n+1}. As Milnor suggests, l_j can be taken to be the link of σ_j in the star neighborhood of σ_j. l_j intersects in some cyclic order the n-simplices of K^n which have σ_j as a face. Suppose $(\tau_{j,1}, \cdots, \tau_{j,m_j})$ is the cyclic order in which l_j intersects the n-simplices of K^n having σ_j as a face, and suppose the intersection number of l_j with $\tau_{j,i}$ is $\epsilon(j, i)$. Let R denote the smallest normal subgroup of F containing the words $(\prod_{i=1}^{m_j} \tau_{j,i}^{\epsilon(j,i)})$, $j = 1, 2, \cdots, q$. Denote F/R by $G(K^n, M^{n+1})$. We call $G(K^n, M^{n+1})$ the In-Group of the imbedding $K^n \subset M^{n+1}$. It is also possible to define $G(K^n, M^{n+1})$ as $\pi_1(M^{n+1})$ modulo the smallest normal subgroup generated by the image of $\pi_1(M^{n+1}-K^n)$ in $\pi_1(M^{n+1})$. The In-Group does not depend on the orientation of M^{n+1}, the orientations of the simplices of K^n, or subdivisions of either.

THEOREM 1. *If $M^{n+1}-K^n$ is connected there is a surjection, α, from $\pi_1(M^{n+1})$ to $G(K^n, M^{n+1})$.*

It is not difficult to see how one may compute all the *possible* In-Groups that a finite n-complex may have. This may be done by assuming in turn all possible distinct cyclic orderings of the n-simplices incident along each $(n-1)$-simplex. Each of these gives a candidate for an In-Group. The collection of these candidates may be called the Out-Groups of the complex.

Then as a corollary to Theorem 1 we have

COROLLARY 1. *A necessary condition for the semi-linear imbedding of an n-complex K^n in a closed orientable manifold M^{n+1} so that $M^{n+1}-K^n$ is connected is that some Out-Group of K^n be a homomorph of $\pi_1(M^{n+1})$.*

As sample applications of this corollary we have verified the following simple statements.

(a) The disjoint union of s closed orientable n-manifolds may be semi-linearly imbedded in a closed orientable $(n+1)$-manifold M^{n+1} without separating M^{n+1} only if $\pi_1(M^{n+1})$ may be mapped onto a free group of rank s. If the n-manifolds are nonorientable, then there must exist a map of $\pi_1(M^{n+1})$ onto the free product of s copies of Z_2.

(b) The 2-complex obtained by identifying the faces of an octahedron so as to obtain a spine of the octahedral space (see Seifert-Threlfall, *Lehrbuch der Topologie*, p. 213) may be semi-linearly imbedded in a closed orientable 3-manifold M^3 without separating M^3 only if $\pi_1(M^3)$ may be mapped onto Z_3.

The techniques of proof of these results are not difficult. In addition to the theorem stated, it is possible to utilize the In-Group to make a combinatorial construction of covering spaces for a nonseparating n-complex in an $(n+1)$-manifold. Some details of proof will appear in an Annals of Mathematics Studies Publication, and others in a forthcoming paper.

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