NOETHERIAN SIMPLE RINGS

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THEOREM 1. *A right noetherian simple ring R with identity is isomorphic to the endomorphism ring of a unital torsion-free module M of finite rank over an integral domain.*

Since any right artinian ring with identity is right noetherian, this theorem generalizes the classical Wedderburn-Artin theorem which states that a right artinian simple ring with identity is (in the sense of isomorphism) the ring of endomorphisms of a unital module V over a (not necessarily commutative) field D.

The implications of this theorem for the structure of R are not yet apparent to the author. For instance, does it imply that R must contain a nontrivial idempotent, if R is not an integral domain?

The conclusion of Theorem 1 holds for any simple ring R with identity which satisfies the maximum conditions on annihilator right ideals and complement right ideals. According to Goldie [2] R will then have a classical right quotient ring R which is a simple artinian ring, that is, a full ring D_n of $n \times n$ matrices over a field D. Actually, we prove the theorem in the following setting.

THEOREM 2. *If R is a simple ring with identity which contains a minimal complement ($=$closed$=$uniform) right ideal, then R is the endomorphism ring of a unital torsion-free module over an integral domain.*

OUTLINE OF THE PROOF. By a theorem of Utumi [3], the maximal right quotient ring S of R is a full ring of l.t.'s in a right vector space over a field D. It is easily checked that there exists a primitive idempotent $e \in S$ such that $K = eSe \cap R \neq 0$. Then $D = eSe$ is a field, $V = Se$ is a right vector space over D, and S is naturally isomorphic to $\Omega = \mathrm{Hom}_D(V, V)$ under a map ϕ which assigns to each $s \in S$ the element $\phi(s)$ which satisfies $\phi(s)x = sx$, for all $x \in V$. Since D is the right quotient field of K (Faith and Utumi [1]), then D is the right quotient field of the subring Δ generated by K and e. Furthermore, $M = Se \cap R$ is a unital torsion-free module over Δ and it can be shown that $V = MD = \{xd^{-1} \mid x \in M, 0 \neq d \in \Delta\}$. Therefore any element γ in $\Gamma = \mathrm{Hom}_\Delta(M, M)$ has a unique extension γ' in Ω. The natural isomorphism $S \cong \Omega$ implies that Γ is isomorphic to the subring $T = \{s \in S \mid sM \subseteq M\}$. Since T contains R, in order to establish

$R \cong \mathrm{Hom}_\Delta(M, M)$, it remains only to show that $T \subseteq R$. Simplicity of R implies $MR = R$, and then

$$R = MR = (TM)R = T(MR) = TR \supseteq T,$$

that is, $T \subseteq R$.

When S is the classical quotient ring of R, then Goldie's theorems imply that V is finite dimensional over D, and then M will have finite rank over Δ.

We state the following corollaries without comment.

COROLLARY 3. *Let R be a simple ring with identity containing an idempotent $e \neq 0$, such that $\Delta = eRe$ is an integral domain. Then if Δ has a right quotient field, e.g., if R is right noetherian, then $R \cong \mathrm{Hom}_\Delta\ (Re, Re)$.*

R is a *right order* in S in case S is a classical right quotient ring of R.

COROLLARY 4. *Let $S = \mathrm{Hom}_D(V, V)$, where V is a right vector space over D, and let R be a right order in S. Then, if either R is a simple ring with identity, or if R is a maximal right order in S, then there exist a right order Δ of D containing an identity, and a Δ-submodule M of V such that R is naturally isomorphic to* $\mathrm{Hom}_\Delta(M, M)$.

REFERENCES

1. Carl Faith and Yuzo Utumi, *On noetherian prime rings*, Trans. Amer. Math. Soc. (1964) (to appear).
2. A. W. Goldie, *Semiprime rings with maximum condition*, Proc. London Math. Soc. (3) **10** (1960), 201–220.
3. Yuzo Utumi, *On quotient rings*, Osaka Math. J. **8** (1956), 1–18.

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