THE EQUIVALENCE OF THE ANNULUS CONJECTURE AND THE SLAB CONJECTURE[1]

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Communicated by M. L. Curtis, May 29, 1964

In [1], the author showed that the Slab Conjecture implies the Annulus Conjecture.

The purpose of this paper is to show that the Annulus Conjecture implies the Slab Conjecture for $n>3$ and hence the two conjectures are equivalent for $n>3$.

R^n, S^n will denote n-space and the n-sphere, respectively. A k-manifold N is embedded in a locally flat manner in an n-manifold M provided each point of N has a neighborhood U in M such that $(U, U\cap N)\approx(R^n, R^k)$.

The Annulus Conjecture. Let S_1^{n-1}, S_2^{n-1} be disjoint locally flat $(n-1)$-spheres embedded in S^n and let M be the submanifold of S^n bounded by $S_1^{n-1}\cup S_2^{n-1}$. Then $M\approx S^{n-1}\times[0, 1]$.

The Slab Conjecture. Let R_1^{n-1}, R_2^{n-1} be disjoint locally flat $n-1$ spaces embedded as closed subsets of R^n and let M be the submanifold of R^n bounded by $R_1^{n-1}\cup R_2^{n-1}$. Then $M\approx R^{n-1}\times[0, 1]$.

THEOREM. *The Annulus Conjecture implies the Slab Conjecture for* $n>3$.

PROOF. Let R_1^{n-1}, R_2^{n-1} be disjoint locally flat $n-1$ spaces embedded as closed subsets of R^n, $n>3$, and let M be the submanifold of R^n bounded by $R_1^{n-1}\cup R_2^{n-1}$. Let $S^n=R^n\cup\{p\}$ be the one-point compactification of R^n and $S_i^{n-1}=R_i^{n-1}\cup\{p\}$ for $i=1, 2$. By the corollary to Theorem 2 of [2], S_i^{n-1} is flat for $i=1, 2$. Hence, we may assume that $S_1^{n-1}=S^{n-1}$, that S_2^{n-1} lies in the northern hemisphere of $S^n=$ the suspension of S^{n-1}, and that $S_1^{n-1}\cap S_2^{n-1}=\{p\}$.

Let B^{n-1} be the unit ball in $S_1^{n-1}=S^{n-1}$ with center p, $r=$ the south pole of S^n, $q=$ the midpoint of the line segment joining p to r in S^n, $L=$ the line segment joining p to q in S^n, and B_r^n, $B_q^n=$ the cones (n-balls) in S^n with bases B^{n-1} and cone points r, q respectively. (See Figure 1.) Now, let $S_3^{n-1}=[S_1^{n-1}\cup\dot{B}_q^n]-\text{Int}(B^{n-1})$. Then S_3^{n-1} is a flat $n-1$ sphere in S^n and $S_3^{n-1}\cap S_2^{n-1}=\emptyset$. By the Annulus Conjecture, $M\cup B_q^n=A^n$ is an n-annulus. We will complete the proof by showing that $M\cup\{p\}$ is homeomorphic to the decomposition space A^n/L and applying Lemma 3 of [3].

[1] This work supported in part by NSF GP-211.

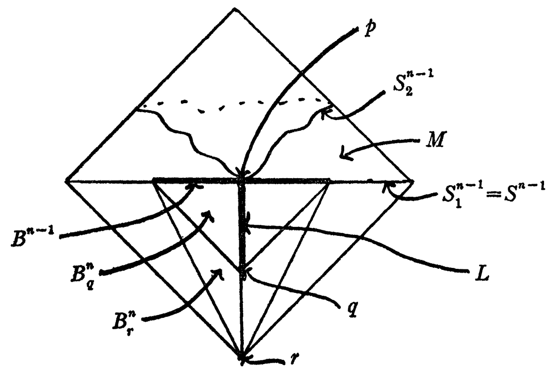

FIGURE 1

By Theorem II.3 of [1], $M_2 = M - R_2^{n-1} \approx R^{n-1} \times [0, 1)$ under some homeomorphism h. Take $T = h^{-1}[h(B^{n-1} - p) \times [0, \frac{1}{2}]]$, $T_r = T \cup B_r^n$, and $T_q = T \cup B_q^n$. Then T_r, T_q are n-balls with $T_q \subset T_r$.

There is a natural map f of T_r onto itself such that the following hold:

(1) $f | \dot{T}_r = 1$,

(2) $f | T_r - L$ is a homeomorphism,

(3) $f(L) = p$,

(4) $f[\mathrm{CL}(B_r^n - B_q^n)] = B_r^n$.

f is obtained by pushing B_q^n up into $T \cup \{p\}$ making use of the parameterization induced on T by $h^{-1} \cdot f$ extends to a map of S^n onto itself by $f | S^n - T_r = 1$.

Since $f(A^n) = M \cup \{p\}$, $f | A^n - L$ is a homeomorphism and $f(L) = p$, it follows that $M \cup \{p\} \approx A^n/L$. By Lemma 3 of [3], since L is a flat arc in A^n with endpoints $p \in S_2^{n-1}$, $q \in S_3^{n-1}$ and $L - (p \cup q) \subset \mathrm{Int}\, A^n$, A^n/L is a pinched annulus, that is, A^n/L is homeomorphic to the one-point compactification of $R^{n-1} \times [0, 1]$. Thus $M \approx R^{n-1} \times [0, 1]$ and the theorem is proved.

COROLLARY. *The Annulus Conjecture is equivalent to the Slab Conjecture for* $n > 3$.

REFERENCES

1. Charles Greathouse, *Locally flat strings*, Bull. Amer. Math. Soc. **70** (1964), 415–418.

2. J. C. Cantrell, *Almost locally flat embeddings of* S^{n-1} *in* S^n, Bull. Amer. Math. Soc. **69** (1963), 716–718.

3. ———, *Some relations between the Annulus Conjecture and union of flat cells theorems* (to appear).

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