MINIMAL SETS AND ERGODIC MEASURES IN $\beta N-N$

BY RALPH A. RAIMI[1]

Communicated by W. Rudin, April 24, 1964

If $t(n)=n+1$ for $n\in N$ (N is the positive integers, βN its Stone-Čech compactification), then t extends uniquely to a continuous mapping (again called t) of βN into βN, and the restriction of t to N^* $=\beta N-N$ is a homeomorphism of N^* onto N^* [**R**, Theorem 4]. If $f\in C(\beta N)$ (the space of continuous real-valued functions on βN), let f_t be determined by $f_t(n)=f(n+1)(n\in N)$. Then for $\omega\in N^*$, $t(\omega)$ is characterized by the relation $f(t(\omega))=f_t(\omega)(f\in C(\beta N))$.

THEOREM. *Every t-invariant, compact, nonempty set $S\subset N^*$, which is minimal with respect to these properties, is the support of at least two ergodic t-invariant Borel probability measures.*

To say that μ is ergodic is to say that every t-invariant Borel set A in N^* has $\mu(A)=1$ or $\mu(A)=0$. We notice that [**BH**, Theorem 2] in our context says:

LEMMA. *If W is the set of all t-invariant Borel probability measures on S, the extreme points of W are exactly the ergodic measures on S.*

PROOF OF THEOREM. Let S be a minimal set, let $\omega\in S$ and put $T_nf=(1/n)\sum_{i=1}^{n}f(t^i\omega)$ $(f\in C(\beta N);\, n=1, 2, \cdots)$. Each T_n is a positive linear functional of norm 1. The set $\{T_n\}$ has at least two limit points in the weak-* topology of the dual space of $C(\beta N)$, for otherwise the sequence T_nf would converge for every $f\in C(\beta N)$, denying Theorem 6 of [**R**].

Let L_1 and L_2 be limit points of $\{T_n\}$. By the Riesz representation theorem, there exist Borel measures μ_1 and μ_2 on βN such that $L_if=\int f\,d\mu_i$ $(i=1, 2;\, f\in C(\beta N))$; these are probability measures because each L_i is positive and of norm 1, and they are t-invariant because $L_if_t=L_if$ $(i=1, 2;\, f\in C(\beta N))$. Furthermore, they are carried by S because $T_nf=1$ for every $f\in C(\beta N)$ whose value at all points of S is 1, and for all n; hence $L_if=1$ for such f $(i=1, 2)$.

Thus the set W of all t-invariant probability measures on S has at least two points. Since W is convex and weak-* compact, the Krein-Milman theorem implies that W has at least two extreme points. By the Lemma, each of these is ergodic. Finally, S is the support of

[1] Work supported by NSF Grant G 23799.

every $\mu \in W$, including these ergodic ones, because S is minimal and every $\mu \in W$ is t-invariant, having therefore an invariant support set.

References

[**BH**] J. R. Blum and L. Hanson, *On invariant probability measures*. I, Pacific J. Math. **10** (1960), 1125–1129.

[**R**] W. Rudin, *Averages of continuous functions on compact spaces*, Duke Math. J. **25** (1958), 195–204.

University of Rochester

HOMOGENEOUS NONNEGATIVE SYMMETRIC QUADRATIC TRANSFORMATIONS

BY G. R. BLAKLEY

Communicated by J. Doob, May 19, 1964

Some recent work [2], [3] has led to almost enough knowledge about nonnegative symmetric homogeneous quadratic transformations to merit the name theory. This note presents one interesting fact, Theorem 6, which states that in a sense almost all such transformations $\mathfrak{I}$ give rise to a sequence $\{\mathfrak{I}^0, \mathfrak{I}^1, \mathfrak{I}^2, \cdots\}$ of iterates which converges pointwise, together with a map of the way stations leading to it. There are very few proofs of the intermediate results since, taken in their totality, they are the skeleton of the proof of Theorem 6. The articulation of this skeleton is indicated by the following scheme of dependences of theorems and lemmas.

T1	on	L1, L2.
T2	on	L3, L4.
T3	on	T1, L4.
T4	on	T2.
T5	on	T1.
T6	on	T1, T3, T4, T5.

Let P be the set of probability n-vectors in Euclidean n-space R^n for some integer $n \geqq 2$ fixed throughout the discussion. P^0 is the set of componentwise positive probability n-vectors and $\partial P = P - P^0$. Let Γ be a symmetric entrywise nonnegative n-by-n matrix and de-