A SHARP FORM OF THE VIRIAL THEOREM[1]

BY HARRY POLLARD

Communicated by Maurice Heins, April 8, 1964

In its classical form the Virial Theorem concerns the behavior of a system S of n mass particles acting under Newtonian attraction in such a fashion that the center of mass 0 remains fixed and the potential energy V satisfies $V > -\infty$ for all positive time t. The latter condition, which is not always stated explicitly, guarantees the analyticity of the coordinates of the particles in the independent variable t; in particular, it excludes collisions [2, pp. 324 ff.]. Let T denote the kinetic energy and h the (constant) total energy $T+V$. Let $\hat{V}$ denote the time average

$$\hat{V} = \lim_{t\to\infty} \frac{1}{t} \int_0^t V(\tau)\, d\tau$$

if the limit exists, with an analogous definition of $\hat{T}$. Clearly each of $\hat{V}$, $\hat{T}$ exists if the other does and $\hat{T}+\hat{V}=h$. The usual theorem states that if S is bounded, in the sense that distances between particles and the velocities of the particles remain bounded, then $\hat{T}$ and $\hat{V}$ exist and $2\hat{T} = -\hat{V}$. An equivalent conclusion is

$$\hat{T} = -h. \tag{1}$$

In this form the theorem is mathematically unsatisfactory because the condition of boundedness is far from necessary. This is already demonstrated by the parabolic case $h=0$ of the two-body problem, $n=2$. In this case distance grows like $t^{2/3}$, so that V behaves like $-t^{-2/3}$ as $t\to\infty$. Consequently, $\hat{V}=0$. Hence $\hat{T}=0$, which is consistent with (1).

We shall replace boundedness by a condition which is both necessary and sufficient. Let $r_{jk}(t)$ denote the distance between particle j and particle k at time t, and let $R(t) = \max_{j,k} r_{jk}(t)$.

THEOREM 1. (1) *is true if and only if*

$$R(t) = o(t), \qquad t \to \infty. \tag{2}$$

Let $2I$ denote the moment of inertia of the system with respect to 0. We begin by showing that (2) is equivalent to

[1] Presented at the Summer Institute for Dynamical Astronomy, Yale University, July 24, 1962.

$$I(t) = o(t^2), \qquad t \to \infty. \tag{3}$$

Since [2, p. 243]

$$2IM = \sum_{1 \le j < k \le n} m_j m_k r_{jk}^2, \tag{4}$$

where M is the total mass, it follows from the definition of R that $I < C_1 R^2$, where C_1 is independent of time. On the other hand, according to (4),

$$2IM > mm' \sum r_{jk}^2,$$

where m, m' are the two smallest masses. At each instant of time $R(t)$ *is* one of the r_{jk}. Therefore, $2IM > mm'R^2$, so $C_2R^2 < I$ where C_2 is a positive constant independent of time. Hence

$$C_2 R^2 < I < C_1 R^2,$$

which implies the equivalence of (2) and (3).

From an integration it is clear that the relation

$$\ddot{I}(t) = o(t), \qquad t \to \infty, \tag{5}$$

implies (3). That it is implied by (3) is more subtle. According to the Lagrange identity [2, p. 235],

$$\ddot{I} = T + h. \tag{6}$$

Since $T \geqq 0$, $\ddot{I} \geqq h > -\infty$. By a theorem of Landau (see, for example [1, Theorem $1B_2$, p. 638]) this property of $\ddot{I}$ entitles us to differentiate each side of (3) to obtain (5). We have proved the equivalence of (2), (3), (5).

Now integrate both sides of (6) and divide by t. Then

$$\frac{\dot{I}(t)}{t} = \frac{1}{t}\int_0^t T(\tau)d\tau + h + O\left(\frac{1}{t}\right), \qquad t \to \infty. \tag{7}$$

It follows that (5) and (1) are equivalent. Hence so are (1) and (2). The theorem stands established.

The next theorem holds with no a priori assumptions on the growth of S.

THEOREM 2. *We have*

$$\limsup_{t \to \infty} \frac{1}{t}\int_0^t T(\tau)d\tau \geqq |h|.$$

Denote the left-hand side by L. Since $T = -V + h$ and $V \leqq 0$ it follows that $T \geqq h$. Hence $L \geqq h$. On the other hand, according to (7)

$$\limsup_{t\to\infty} \frac{\dot{I}(t)}{t} = L + h.$$

I claim $L+h \geqq 0$; otherwise $\dot{I}(t) \leqq -\epsilon t$ for large t, where $\epsilon > 0$. Integrating and dividing by t^2 yields

$$\limsup_{t\to\infty} \frac{I(t)}{t^2} \leqq -\epsilon,$$

which is impossible since $I \geqq 0$. Hence $L \geqq h$, $L \geqq -h$, from which the theorem follows.

COROLLARY. *If* $\hat{T}=0$, *then* $h=0$ *and* $R(t)=o(t)$.

The first conclusion follows from Theorem 2. Then the second follows from Theorem 1.

I am indebted to Professor M. Golomb for observing that my original proof of the Corollary actually proves the stronger Theorem 2.

REFERENCES

1. R. P. Boas, *Asymptotic relations for derivatives*, Duke Math. J. **3** (1937), 637–646.

2. A. Wintner, *The analytical foundations of celestial mechanics*, Princeton Univ. Press, Princeton, N. J., 1942.

PURDUE UNIVERSITY