NONLOCAL ELLIPTIC BOUNDARY VALUE PROBLEMS

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1. **Introduction.** Let A be an elliptic operator on a region G. Boundary value problems of the form $Au=f$ on G, $Bu=0$ on the boundary ∂G, have been studied extensively for the case in which B is a system of differential operators; see, for example, [1], [4], [5], [9]. Bade and Freeman [3] have obtained results for a class of nonlocal problems, i.e., problems in which the operators B are not necessarily differential operators. In [3], A is taken to be the Laplace operator and B is of the form $Bu=\partial u/\partial n-Cu$, where $\partial/\partial n$ is the normal derivative and C is any bounded operator in $L^2(\partial G)$.

In this note we indicate some extensions of the results of [3] to general classes of nonlocal boundary value problems for elliptic operators of arbitrary even order with variable coefficients. Details and proofs will appear elsewhere. This research is part of the author's doctoral dissertation, prepared under the direction of Professor Felix Browder at Yale University. The author is grateful to Professor Browder for his advice and encouragement.

2. **Main results.** For the definition of various spaces of functions and distributions we refer to [7, §1]. The operators considered are all linear, and we denote the domain and range of an operator T by $D(T)$ and $R(T)$, respectively.

Let A be an elliptic operator of order $2p$ defined on a region $G\subseteq E^n$, and let $(B_0, B_1, \cdots, B_{2p-1})$ be a system of differential operators defined near the boundary ∂G. The boundary value problems to be considered here are of the form

$$(*)\qquad Au=f,\qquad B_k u=\sum_{j\in J} C_{kj}B_j u,\qquad k\in K,$$

where (C_{kj}) is a system of operators defined in suitable function spaces on ∂G and the index sets J and K are complementary subsets of $\{0, 1, \cdots, 2p-1\}$.

Let A_1 be the maximal operator for A, i.e., the operator in $L^2(G)$ which is the restriction of A to those functions u in $L^2(G)$ such that the distribution Au is in $L^2(G)$. We wish to study the restriction of A_1 to those functions satisfying in some sense the above boundary conditions. This can be accomplished by studying perturbations of

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an associated operator S acting in the direct sum of $L^2(G)$ and a boundary space. For second-order operators this approach is due to Calkin and is carried out in [3].

THEOREM 1. *Let G be a region in E^n which is uniformly regular of class $4p$. Let the operators $(A, B_0, B_1, \cdots, B_{2p-1})$ and $(A', B_0', B_1', \cdots, B_{2p-1}')$ satisfy the conditions* (i)–(v) *below with respect to the index sets J, K, J', K'. Let S_0 be the operator*

$$S_0\colon L^2(G) \oplus \sum_{j\in J} W^{-j,2}(\partial G) \to L^2(G) \oplus \sum_{j\in J'} W^{j,2}(\partial G)$$

having domain $D(S_0)=\{[u, ((-1)^jB_ju|j\in J)]\,|\,u\in C_0^\infty(\overline{G})\}$ and with $S_0[u, ((-1)^jB_ju)]=[Au, (B_ku|k\in K)]$. Let S_0' be defined similarly. Then:

(a) *S_0 and S_0' are closable, and their closures S and S' are adjoints.*

(b) *Let Ω be the set of all u in $D(A_1)$ such that the distribution derivatives of u of order $\leqq 2p-1$ have L^2 boundary values on ∂G. Then $D(S) = \{[u, ((-1)^jB_ju)]\,|\,u\in\Omega\}$, and similarly for S'.*

(c) *Suppose G is bounded. Then S and S' are Fredholm operators.*

See [6, Definition 1] for the definition of uniform regularity and [3] for the definition of L^2 boundary values. The conditions on the operators in Theorem 1 are:

(i) A and A' are operators of order $2p$ defined on a neighborhood of the closure of G, whose coefficients have bounded continuous derivatives of order $\leqq 2p$; A and A' are uniformly elliptic and regularly elliptic on G [6, Definition 5].

(ii) B_k and B_k' are differential operators of order k defined on a neighborhood of ∂G and having coefficients with bounded continuous derivatives of order $<2p$; the surface ∂G is uniformly noncharacteristic for B_k and B_k'.

(iii) The systems (A, B) and (A', B') are *conjugate* in the sense that for all u, v in $C_0(\overline{G})$,

$$(1) \qquad (Au, v) - (u, A'v) = \sum (-1)^{k+1}(B_{2p-1-k}u, B_k'v).$$

Here the parentheses on the left denote the inner product in $L^2(G)$ and those on the right denote the inner product in $L^2(\partial G)$. (If the system (A, B) satisfies conditions (i) and (ii), then there is a system (A', B') satisfying (i) and (ii) and such that (1) holds; see [2].)

(iv) J is an ordered subset of $\{0, 1, \cdots, 2p-1\}$ consisting of p elements in decreasing order, K is the complementary subset in increasing order, $J'=(2p-1-k|k\in K)$, and $K'=(2p-1-j|j\in J)$.

(v) The systems $(A, (B_k|k\in K))$ and $(A', (B_k'\,|k\in K'))$ are regular [5, Definition 1].

For convenience we denote the spaces on ∂G by

$$E_0 = \sum_{j\in J} W^{-j,2}(\partial G), \qquad E_1 = \sum_{j\in J'} W^{j,2}(\partial G),$$

and similarly for E_0' and E_1'. We denote the systems of operators $((-1)^j B_j | j \in J)$ by γ_0 and $(B_k | k \in K)$ by γ_1, and similarly for γ_0' and γ_1'. The system (*) above is then $Au=f$, $(\gamma_1 - C\gamma_0)u=0$, where $C: E_0 \to E_1$. The corresponding realization of the operator A is $A(C)$, the restriction of A_1 to $\{u | u\in\Omega,\ (\gamma_1 - C\gamma_0)u=0\}$. If we define the *induced operator* $C_1: L^2(G) \oplus E_0 \to L^2(G) \oplus E_1$ by $C_1[u, f]=[0, Cf]$, $u\in L^2(G)$, $f\in D(C)$, then the corresponding perturbation of S is $S-C_1$. The following theorems are derived by exploiting the connections between properties of S, properties of perturbations $S-C_1$, and properties of $A(C)$.

THEOREM 2. *Let G, A, B, A', B' be as in Theorem 1. Let $C: E_0\to E_1$ be such that $D(C)\supseteq \gamma_0(\Omega)$ and such that for some positive constants $\epsilon<1$ and k the inequality*

$$\|C\gamma_0 u\| \leqq \epsilon \|S[u, \gamma_0 u]\| + k\|u\|$$

holds for all $u\in\Omega$. Let $A(C)$ be the restriction of the maximal operator A_1 to $\{u | u\in\Omega,\ (\gamma_1 - C\gamma_0)u=0\}$. Then:

(a) *$A(C)$ is closed.*

(b) *If in addition $R(C)\subseteq \sum W^{k+1/2,2}(\partial G)(k\in K)$, then the domain of $A(C)$ is contained in $W^{2p,2}(G)$.*

Let X and Y be Banach spaces, $T: X\to Y$. An operator $C: X\to Y$ is said to be *T-compact* if C is closable, $D(C)\subseteq D(T)$, and C is compact as an operator from $D(T)$ to Y with respect to the graph topology on $D(T)$.

THEOREM 3. *Let G, A, B, A', B' be as in Theorem 1. Let $C: E_0\to E_1$ and $C': E_0'\to E_1'$ be such that the induced operators C_1 and C_1' are S-compact and S'-compact, respectively. Then:*

(a) *The operators $A(C)$ and $A'(C')$ are closed.*

(b) *If $(S-C_1)^* = S'-C_1'$, then $A(C)$ and $A'(C')$ are adjoints.*

(c) *If G is bounded, then $A(C)$ and $A'(C')$ are Fredholm operators with the same indices as S and S', respectively. Moreover if $C'\subseteq C^*$, then the condition in* (b) *is necessarily satisfied.*

As a special case, suppose A is a second order operator of the form $A=\sum D_j(a_{jk}D_k)$, where $D_j = i^{-1}(\partial/\partial x_j)$ and $a_{jk}=a_{kj}$. As γ_1 we take the operator $-i\sum n_j a_{jk} D_k$, where n is the unit inner normal vector to ∂G; call this the *Neumann operator* for A. Let $\gamma_0 u=\gamma_0' u$

$=u|\partial G$ and let γ_1' be the Neumann operator for the formal adjoint A'. In this case $J=\{0\}$, so $E_0=E_1=E_0'=E_1'=L^2(\partial G)$.

THEOREM 4. *Let $A=\sum D_j(a_{jk}D_k)$ with $a_{jk}=a_{kj}$, let γ_1 and γ_1' be the Neumann operators for A and its formal adjoint A', and let $\gamma_0 u=\gamma_0' u=u|\partial G$. Assume that G is uniformly regular of class $4p$, that the a_{jk} have bounded continuous derivatives of order $\leqq 2$ on $\overline{G}$, and that A is uniformly strongly elliptic on G. Let A_1 and Ω be defined as above. Then if C is any bounded operator in $L^2(\partial G)$ and $A(C)$ is the restriction of A_1 to $\{u|u\in\Omega, \gamma_1 u=Cu\}$, the operator $A(C)$ is closed and its adjoint is $A'(C^*)$, where this is defined similarly. Furthermore the spectrum of $A(C)$ is contained in a triangular region of the form $\{\lambda|\operatorname{Re}(\lambda)\geqq -a+b|\operatorname{Im}(\lambda)|\}$, where a and b are constants depending on $\|C\|$*

REMARKS. 1. The analytical tools used in [**3**] for connecting properties of realizations of A with those of perturbations of S are potential-theoretic in nature. In the general case this role is played by certain a priori estimates; the basic estimate shows that the set Ω above is contained in $W^{2p-1/2,2}(G)$, hence all the realizations $A(C)$ obtained above have domains in this space.

2. Theorem 2 (b) answers affirmatively a question raised in [**3**].

3. Theorem 4 also holds for certain classes of unbounded operators C. If $A=A'$ in this theorem, then the spectrum of $A(C)$ for C bounded is contained in a parabola opening to the right; this is proved in [**3**] for the case of a bounded region G.

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