THE SU-BORDISM THEORY

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To A. D. Wallace on his sixtieth birthday

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1. **Introduction.** The object of this note is a discussion of the special unitary (or SU-) bordism theory. Specifically we shall show that the SU-bordism groups of a point, which are the coefficient groups for this generalized homology theory, have no odd torsion and contain no elements of order four. The absence of odd torsion was first established by Novikov [4], who also showed that these groups in odd dimensions have no elements of order four.

Our approach is parallel to Atiyah's use of bordism to study the relation of oriented to unoriented cobordism [1]. We shall need the Milnor spectra MU and MSU [3]. We take $MU(n)$, respectively $MSU(n)$, to be the Thom space of the universal bundle $\eta_n \to BU(n)$, respectively $\xi_n \to BSU(n)$. The natural maps $S^2 \wedge MU(n) \to MU(n+1)$, and $S^2 \wedge MSU(n) \to MSU(n+1)$ give rise to the spectra.

The SU-bordism groups of a finite simplicial pair are defined by

$$\Gamma_k(X, A) = \pi_{k+2n}(MSU(n) \wedge X/A), \qquad n \text{ large}.$$

This is the homology theory associated to the spectrum MSU by G. W. Whitehead [6]. Similarly $\mathfrak{U}_k(X, A) = \pi_{k+2n}(MU(n) \wedge X/A)$, n large. We shall recall that $X/\varnothing = X \cup \infty$, the disjoint union of X with a point.

2. **The basic isomorphisms.** We establish in this section two isomorphisms relating SU-bordism to U-bordism. We let $\alpha \to BU(1)$ be a $U(m)$-bundle over the classifying space of $U(1)$, and we let $(M(\alpha), \infty)$ be the associated Thom space with base point at infinity.

(2.1) *If the first Chern class of α is a generator of $H^2(BU(1); Z)$ then there is a canonical isomorphism $\tilde{\Gamma}_{k+2m}(M(\alpha) \wedge X/A) \simeq \mathfrak{U}_k(X, A)$ for every finite* CW-*pair.*

Briefly there is a bundle map

$$\begin{array}{ccc} \xi_n \times \alpha & \xrightarrow{\;F\;} & \eta_{n+m} \\ \downarrow & & \downarrow \\ BSU(n) \times BU(1) & \xrightarrow{\;f\;} & BU(n+m) \end{array}$$

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for which $f^*: H^i(BU(n+m); Z) \simeq H^i(BSU(n) \times (BU(1); Z))$, $0 \leqq i \leqq 2n+1$. In view of the commutative Thom diagram [5], F^*: $H^i(M(\eta_{n+m}), \infty; Z) \simeq H^i(M(\xi_n \times \alpha), \infty; Z)$, $0 \leqq i \leqq 4n+2m+1$. For each finite CW-pair it follows that

$$(F \times \mathrm{id})_*: \pi_{k+2(m+n)}(M(\xi_n \times \alpha) \wedge X/A) \simeq \pi_{k+2(m+n)}(M(\eta_{n+m}) \wedge X/A)$$

for $k \leqq 2n$. However $M(\xi_n \times \alpha) = MSU(n) \wedge M(\alpha)$, and $M(\eta_{n+m}) = MU(n+m)$; thus

$$\tilde{\Gamma}_{k+2m}(M(\alpha) \wedge X/A) \simeq \mathfrak{U}_k(X, A).$$

We shall next indicate an immediate corollary. We consider $CP(N) \subset CP(\infty) = BU(1)$ together with the stunted projective space $CP(\infty)/CP(N)$.

(2.2) *For every finite* CW-*pair*

$$\tilde{\Gamma}_{k+4N+2}((CP(\infty)/CP(2N)) \wedge (X/A)) \simeq \mathfrak{U}_k(X, A).$$

The formula is also valid for $N=0$. We let $\eta_1 \to BU(1)$ be the canonical line bundle and $\bar{\eta}_1 \to BU(1)$ be the conjugate bundle. We take $\alpha = (N+1)\eta_1 \oplus N\bar{\eta}_1$. The total Chern class of α is $(1+c)(1-c^2)^N$, hence (2.1) applies to α. We must show that the Thom space of α is $CP(\infty)/CP(2N)$. For this, however, we may as well consider the Thom space of $(2N+1)\eta_1$, which is real equivalent to α. Now $CP(n-2N-1) \subset CP(n)$ with normal bundle the restriction of $(2N+1)\eta_1$. The Thom space of this normal bundle is $CP(n)/CP(2N)$, so we let $n \to \infty$ and $CP(\infty)/CP(2N)$ is indeed the Thom space of $(N+1)\eta_1 \oplus N\bar{\eta}_1$. Note explicitly that $\tilde{\Gamma}_{k+4N+2}(CP(\infty)/CP(2N)) \simeq \mathfrak{U}_k$.

We turn next to an $SU(m)$-bundle $\beta \to BU(1)$ with Thom space $M(\beta)$. An analogous result to (2.1) is

(2.3) *If the second Chern class of β is a generator of $H^4(BU(1); Z)$ then*

$$\tilde{\Gamma}_{k+2m}(M(\beta) \wedge X/A) \simeq \Gamma_k(CP(\infty) \times X, CP(\infty) \times A)$$

for every finite CW-*pair.*

We consider first the bundle map

$$\begin{array}{ccc} \xi_n \times \beta & \xrightarrow{F} & \xi_{n+m} \\ \downarrow & & \downarrow \\ BSU(n) \times BU(1) & \xrightarrow{f} & BSU(n+m) \end{array}$$

and we let $\sigma_{n+m} \to BSU(n+m) \times BU(1)$ be the bundle induced from ξ_{n+m} by projection on the first coordinate. We obtain a new bundle map

$$\begin{array}{ccc} \xi_n \times \beta & \xrightarrow{\quad G \quad} & \sigma_{n+m} \\ \downarrow & & \downarrow \\ BSU(n) \times BU(1) & \xrightarrow{g} & BSU(n+m) \times BU(1) \end{array}$$

where $g(x, y) = (f(x, y), y)$. We argue that

$$(G \times \mathrm{id})_*\colon \pi_{k+2(m+n)}(M(\xi_n \times \beta) \wedge X/A) \simeq \pi_{k+2(m+n)}(M(\sigma_{n+m}) \wedge X/A)$$

for $k+2m \leqq 2n$. Now $M(\xi_n \times \beta) = MSU(n) \wedge M(\beta)$, and $M(\sigma_{n+m}) = MSU(n+m) \wedge (CP(\infty)/\emptyset)$, hence

$$\tilde{\Gamma}_{k+2m}(M(\beta) \wedge X/A) \simeq \tilde{\Gamma}_k((CP(\infty)/\emptyset) \wedge (X/A)).$$

Finally $\tilde{\Gamma}_k((CP(\infty)/\emptyset) \wedge (X/A)) = \Gamma_k(CP(\infty) \times X, CP(\infty) \times A)$. An immediate corollary is

(2.4) *For every finite* CW-*pair*

$$\tilde{\Gamma}_{k+4}((CP(\infty)/CP(1)) \wedge (X/A)) \simeq \Gamma_k(CP(\infty) \times X, CP(\infty) \times A).$$

We take $\beta = \eta_1 \oplus \bar{\eta}_1 \rightarrow CP(\infty)$ and apply (2.3). To use (2.4) we need one more relation

(2.5) *For every finite* CW-*pair*

$$\Gamma_k(CP(\infty) \times X, CP(\infty) \times A) \simeq \Gamma_k(X, A) \oplus \mathfrak{U}_{k-2}(X, A).$$

There is a split exact sequence of spaces

$$0 \rightarrow X/A \rightarrow CP(\infty) \times X/CP(\infty) \times A \rightarrow CP(\infty) \wedge X/A \rightarrow 0$$

from which (2.5) follows since $\tilde{\Gamma}_k(CP(\infty) \wedge X/A) \simeq \mathfrak{U}_{k-2}(X, A)$.

3. The exact sequences. We shall combine the isomorphisms into several exact sequences. First consider

$$0 \rightarrow CP(1) \wedge X/A \rightarrow CP(\infty) \wedge X/A \rightarrow (CP(\infty)/CP(1)) \wedge (X/A) \rightarrow 0$$

together with the reduced SU-bordism sequence

$$\cdots \rightarrow \tilde{\Gamma}_{k+2}(CP(1) \wedge X/A) \rightarrow \tilde{\Gamma}_{k+2}(CP(\infty) \wedge X/A)$$
$$\rightarrow \tilde{\Gamma}_{k+2}(CP(\infty)/CP(1) \wedge X/A) \rightarrow \cdots.$$

Since $CP(1) = S^2$, $\tilde{\Gamma}_{k+2}(CP(1) \wedge X/A) \simeq \Gamma_k(X, A)$. We use (2.2), (2.4) and (2.5) to obtain

(3.1) *For every finite* CW-*pair there is an exact sequence*

$$\cdots \rightarrow \Gamma_k(X, A) \rightarrow \mathfrak{U}_k(X, A) \rightarrow \Gamma_{k-2}(X, A) \oplus \mathfrak{U}_{k-4}(X, A)$$
$$\rightarrow \Gamma_{k-1}(X, A) \rightarrow \cdots.$$

This is the analogue of the exact sequence [1]

$$\cdots \to \Omega_k(X, A) \to \mathfrak{N}_k(X, A) \to \Omega_{k-1}(X, A) \oplus \mathfrak{N}_{k-2}(X, A) \to \Omega_{k-1}(X, A) \to \cdots.$$

Now from (2.2) it follows that $\tilde{\Gamma}_{k+2}(CP(\infty)) \simeq \mathfrak{U}_k$. Let us set $\Gamma_k = \Gamma_k(\text{pt.})$, so the Γ_k are the coefficient groups. We know, therefore [5], that

$$\tilde{\Gamma}_{k+2}(CP(\infty)) \otimes Q = \sum_0^{k+2} \tilde{H}_{k+2-j}(CP(\infty) \otimes \Gamma_j \otimes Q) = \mathfrak{U}_k \otimes Q;$$

hence Γ_{2j+1} is finite and rank $\Gamma_{2j} = \text{rank } \mathfrak{U}_{2j} - \text{rank } \mathfrak{U}_{2j-2}$. We are especially concerned with computing the groups $\tilde{\Gamma}_k(CP(2))$. We see immediately that $\tilde{\Gamma}_{2j+1}(CP(2))$ is finite.

The exact sequence $0 \to CP(2) \to CP(\infty) \to CP(\infty)/CP(2) \to 0$ with the aid of (2.2) yields

$$\cdots \to \tilde{\Gamma}_k(CP(2)) \to \mathfrak{U}_{k-2} \to \mathfrak{U}_{k-4} \to \tilde{\Gamma}_{k-1}(CP(2)) \to \cdots.$$

Since $\mathfrak{U}_{2j}$ is free and $\mathfrak{U}_{2j+1} = 0$ we have

$$0 \to \tilde{\Gamma}_{2(j+1)} \to \mathfrak{U}_{2j} \to \mathfrak{U}_{2j-2} \to \tilde{\Gamma}_{2j+1}(CP(2)) \to 0;$$

hence $\tilde{\Gamma}_{2j}(CP(2))$ is free.

Next we wish to apply (3.1) to $(CP(2), \text{pt.})$. From [2, (2.2)] we have $\tilde{\mathfrak{U}}_{2j}(CP(2))$ is free and $\tilde{\mathfrak{U}}_{2j+1}(CP(2)) = 0$. Applying (3.1) we have

$$0 \to \tilde{\Gamma}_{2j+1}(CP(2)) \to \tilde{\Gamma}_{2j+2}(CP(2)) \to \cdots;$$

hence $\tilde{\Gamma}_{2j+1}(CP(2)) = 0$. It follows that $\tilde{\Gamma}_{2(j+1)}(CP(2))$ is a direct summand of $\mathfrak{U}_{2j}$. Actually it is the subgroup of weakly complex cobordism classes for which every Chern number involving c_1^2 vanishes. To complete our argument we turn to $0 \to CP(1) \to CP(2) \to CP(2)/CP(1) \to 0$. Now $\tilde{\Gamma}_{k+2}(CP(1)) \simeq \Gamma_k$ and since $CP(2)/CP(1)$ is S^4,

$$\tilde{\Gamma}_{k+2}(CP(2)/CP(1)) \simeq \Gamma_{k-2}.$$

We have thus

$$\to \Gamma_k \to \tilde{\Gamma}_{k+2}(CP(2)) \to \Gamma_{k-2} \xrightarrow{\theta} \Gamma_{k-1} \to \tilde{\Gamma}_{k+1}(CP(2)) \to \cdots.$$

Since $\tilde{\Gamma}_{2j+1}(CP(2)) = 0$, this becomes

$$0 \to \Gamma_{2j-1} \xrightarrow{\theta} \Gamma_{2j} \to \tilde{\Gamma}_{2(j+1)}(CP(2)) \to \Gamma_{2(j-1)} \xrightarrow{\theta} \Gamma_{2j-1} \to 0.$$

(3.2) *The image of* $\theta: \Gamma_k \to \Gamma_{k+1}$ *consists entirely of elements of order* 2.

We postpone the discussion of this point until the final section. Since $\tilde{\Gamma}_{2(j+1)}$ is free we have

(3.3) *The groups* Γ_k *contain no odd torsion, nor do they contain elements of order four.*

We have been informed that A. Liulevicius has computed a considerable number of the groups Γ_k; the first few are $\Gamma_1 \simeq Z_2$, $\Gamma_2 \simeq Z_2$, $\Gamma_3 = 0$, $\Gamma_4 \simeq Z$.

4. Homomorphisms in a homology. We wish to assign to each element in $\Sigma_p = \pi_{p+q}(S^q)$, $p \leqq q-1$, a stable, natural additive homology operation $\theta: \Gamma_n(X, A) \to \Gamma_{n+p}(X, A)$. We select a map $f: (S^{p+q}, x_0) \to (S^q, y_0)$ and consider

$$(f \times \mathrm{id})_*: \tilde{\Gamma}_{n+p+q}(S^{p+q} \wedge X/A) \to \tilde{\Gamma}_{n+p+q}(S^q \wedge X/A).$$

We define $\theta: \Gamma_n(X, A) \to \Gamma_{n+p}(X, A)$ so that the diagram

$$\begin{array}{ccc} (f \times \mathrm{id})_*: \tilde{\Gamma}_{n+p+q}(S^{p+q} \wedge X/A) & \to & \tilde{\Gamma}_{n+p+q}(S^q \wedge X/A) \\ \uparrow \simeq & & \uparrow \simeq \\ \theta: \Gamma_n(X, A) & \longrightarrow & \Gamma_{n+p}(X, A) \end{array}$$

commutes. The definition only depends on the stable homotopy class of f. Furthermore if $g: S^{p+q} \to S^q$ is another map with operation θ', then $\theta + \theta'$ corresponds to $f+g \in \Sigma_p$. For example, if $f \in \Sigma_1 \simeq Z_2$, then every element in the image of θ has order two.

We may attach a $(p+q+1)$-cell to S^q via f to obtain a space Y. Now there is an exact sequence of spaces

$$0 \to S^q \wedge X/A \to Y \wedge X/A \to S^{p+q+1} \wedge X/A \to 0$$

which gives rise to an exact sequence

$$\cdots \to \Gamma_{n+1}(X, A) \to \Gamma_{n+q+1}(Y \wedge X/A) \to \Gamma_{n-p}(X, A) \xrightarrow{\theta} \Gamma_n(X, A) \to \cdots.$$

This indicates the role of θ in (3.2), for there we took $f: S^3 \to S^2$ to be the Hopf map, so $Y = CP(2)$ and $f \in \Sigma_1 \simeq Z_2$ and every element in the image of θ has order two. Of course the construction of the operators may be carried out in any generalized homology theory.

Added in proof. In their paper SU-*cobordism and the Arf invariant*, Lashof and Rothenberg also showed that Γ_{2k} has no elements of order four.

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A NOTE ON APPROXIMATION BY BERNSTEIN POLYNOMIALS

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Let f be continuous on $[0, 1]$ and $0 \leqq \alpha < \beta \leqq 1$ and let B_nf be the Bernstein polynomial of f of degree n, defined by

$$B_nf(x) = \sum_{\nu=0}^{n} f\left(\frac{\nu}{n}\right)\binom{n}{\nu} x^\nu (1 - x)^{n-\nu}.$$

In view of a result of E. V. Voronovskaya, which states that the boundedness of f on $[0, 1]$ and the existence of f'' at a point $x \in [0, 1]$ implies that

$$B_nf(x) - f(x) = \frac{x(1 - x)}{2n} f''(x) + o\left(\frac{1}{n}\right) \qquad (n \to \infty),$$

it has been conjectured [1, p. 22] that the relation

$$B_nf(x) - f(x) = o\left(\frac{1}{n}\right)$$

cannot be true for all $x \in [\alpha, \beta]$ unless f is a linear function on $[\alpha, \beta]$. The following theorem related to this conjecture was proved by K. de Leeuw [2]:

If f is continuous on $[0, 1]$ and

$$B_nf(x) - f(x) = O\left(\frac{1}{n}\right)$$