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MULTIPLIERS OF p-INTEGRABLE FUNCTIONS[1]

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1. Introduction. Let G be a locally compact Abelian group. Let $L^p(G)$ $(1 \leqq p < \infty)$ be the space of p-integrable functions (with respect to the Haar measure) with the usual norm. A *multiplier* of $L^p(G)$ is a bounded linear operator T of $L^p(G)$ into $L^p(G)$ which commutes with the translation operators; that is, $\tau_y T = T\tau_y$ for all $y \in G$, where $\tau_y f(x) = f(x+y)$. The space of multipliers will be denoted by $M_p = M_p(G)$. It is known that M_1 is isomorphic and isometric to the space of bounded regular Baire measures on G and that M_2 is isomorphic and isometric to $L^\infty(\Gamma)$, where Γ is the character group of G, and thus M_2 is the conjugate space of the space $A(G)$ of continuous functions on G which are Fourier transforms of elements of $L^1(\Gamma)$. Theorem 1 below asserts that, for $1 < p < \infty$, M_p is also the conjugate space of a space A_p of continuous functions on G. A corollary of this fact is that M_p is the closure in the weak operator topology of the linear span of the translation operators. A theorem due to Hörmander relating tempered distributions on R^n to $M_p(R^n)$ [2], is also an easy consequence of Theorem 1. In view of the fact that a multiplier T can be identified with an element $T^\frown \in L^\infty(\Gamma)$ (Γ being the character group of G), another consequence of Theorem 1 is that if $T \in M_p$, $T^\frown * \mu = U^\frown$ with $U \in M_p$, where μ is a bounded regular Baire measure on Γ. If G is a noncommutative unimodular group, a proposition analogous to Theorem 1 holds for operators commuting with right (respec-

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tively, left) translations; the specialization to the case $p=2$ yields results established by Segal in [7]. Theorem 7 relates lacunary subsets of a discrete Abelian group Γ to multipliers on $L^p(G)$ where G is the character group of Γ.

2. **The space of multipliers as a conjugate space.** Hereafter p will be a fixed real number, $1<p<\infty$, and q will be such that $1/p+1/q=1$. $C_{00}(G)$ and $C_0(G)$ will be, respectively, the space of continuous functions with compact support and the space of continuous functions vanishing at infinity on G. The convolution with respect to the Haar measure of two measurable functions f and g will be denoted by $f*g$ whenever it is well defined.

DEFINITION 1. *Let A_p be the space of functions $h\in C_0(G)$ which can be written as $h=\sum_{i=1}^{\infty} c_i f_i * g_i$, where $f_i\in L^p(G)$, $g_i\in L^q(G)$ and $\sum|c_i|\,\|f_i\|_p\|g_i\|_q<\infty$; for $h\in A_p$ define*

$$\|h\|_{A_p} = \inf\left\{\sum_{i=1}^{\infty}|c_i|\,\|f_i\|_p\|g_i\|_q : h=\sum_{i=1}^{\infty} c_i f_i * g_i\right\}.$$

THEOREM 1. *M_p is isometric and isomorphic to the conjugate space of A_p, the element $T\in M_p$ corresponding to the functional $\phi_T(h)=\sum c_i(Tf_i * g_i)(0)$, where $h=\sum c_i f_i * g_i$. The weak operator topology of M_p coincides, on the unit sphere of M_p, with the weak-star topology induced by A_p.*

One notices that if $f, g\in C_{00}(G)$, $T(f*g) = Tf*g$, so that $T(f*g)$ is, after correction on a set of Haar measure zero, a continuous function. One can then define the functional $\phi_T(h)=Th(0)$, on the space S of linear combinations of functions of the type $f*g$ with $f, g\in C_{00}(G)$. Thus the proof of Theorem 1 consists essentially of showing that the completion of S, under the norm

$$\|h\| = \sup\{|Th(0)| : T\in M_p, \|T\|_{M_p}\leqq 1\},$$

is A_p. This is accomplished using the fact that the BX topology on the conjugate space X^* of a Banach space X has the same continuous linear functionals as the X topology (weak star topology) (cf. [1, V, 5.6]).

COROLLARY 2. *M_p is the closure, in the weak operator topology, of the span of the translation operators.*

REMARK 3. *As a consequence of the Riesz convexity theorem* [1, V, 10.11], *and in view of the duality between $L^p(G)$ and $L^q(G)$, the restriction to $C_{00}(G)$ of an element $T\in M_p$ can be extended to an element of M_r for $p\leqq r\leqq q$. Furthermore $\|T\|_{M_p}=\|T\|_{M_q}$ and, if $p\leqq r\leqq s\leqq 2$, $\|T\|_{M_r}\leqq\|T\|_{M_s}$. Dually one has $A_p=A_q$ and, if $p\leqq r\leqq s\leqq 2$, $A_s\subseteq A_r$,* *with* $\|\cdot\|_{A_r} \leqq \|\cdot\|_{A_s}$. One should also notice that $A_2(G)=A(G)$ is the space of Fourier transforms of elements of $L^1(\Gamma)$, where Γ is the character group of G. Thus, since A_2 is dense in A_p, each $T\in M_p$ *corresponds biuniquely to an element* $T^\frown(\gamma)$ of $L^\infty(\Gamma)$. $T^\frown$ will be called the *transform* of T.

REMARK 4. A simple consequence of Theorem 1 and of the fact that $A_2=A(G)$ is continuously and densely embedded in A_p, is the result proved by Hörmander in [2], stating that for $G=\mathbf{R}^n$, M_p can be identified with a subspace of the space of tempered distributions on $\mathbf{R}^n$. It suffices to notice that the space $\mathcal{S}$ of rapidly decreasing functions on $\mathbf{R}^n$ is continuously and densely embedded in $A(\mathbf{R}^n)$ and therefore in A_p (cf., e.g., [3, I, 1.7]).

COROLLARY 5. *Let* $T\in M_p$ *and let* $T^\frown$ *be its transform in the sense of Remark* 3; *then, if* μ *is a bounded regular Baire measure on* Γ (*the character group of* G), $T^\frown * \mu$ *is also the transform of a multiplier in* M_p.

One shows that if $\hat{\mu}$ is the Fourier-Stieltjes transform of μ, $\hat{\mu}h\in A_p$ for every $h\in A_p$; thus the functional $\phi(h)=T(\hat{\mu}h)(0)$ on A_p defines a multiplier whose transform is $T^\frown * \mu$.

REMARK 6. Theorem 1 and Corollary 2 are valid for not necessarily commutative unimodular groups in the following sense: Let $\mathcal{L}_p$ (respectively, $\mathcal{R}_p$) be the space of bounded linear operators on $L^p(G)$ which commute with right (respectively, left) translations by elements of G. Let A_p^1 (respectively, A_p^2) be the space functions on G which can be written as $\sum_{i=1}^\infty c_i f_i * g_i$, with $f_i\in L^p(G)$, $g_i\in L^q(G)$ (respectively, $f_i\in L^q(G)$, $g_i\in L^p(G)$) with f_i, g_i, c_i satisfying the conditions of Definition 1 and with norms analogously defined; then $\mathcal{L}_p$ (respectively, $\mathcal{R}_p$) is the conjugate space of A_p^1 (respectively, A_p^2). Moreover, $\mathcal{L}_p$ (respectively, $\mathcal{R}_p$) is the closure, in the weak operator topology, of the space of the left (respectively, right) translations. Thus the space $\mathcal{R}_p'$ of the operators commuting with elements of $\mathcal{R}_p$ is $\mathcal{L}_p$ and conversely, so that $\mathcal{R}_p\cap\mathcal{L}_p$ is the center of both $\mathcal{L}_p$ and $\mathcal{R}_p$. For the case $p=2$ these results specialize to known results due to Segal [7]. One should also notice that $A_p^1=A_q^2$ and therefore $\mathcal{L}_p$ is isometric and linearly isomorphic to $\mathcal{R}_q$.

3. **Multipliers and lacunary sets.** Let G be a compact Abelian group, Γ its discrete character group. A set $E\subseteq\Gamma$ is called a *Sidon set* (cf. [5, 5.7.2]) if every $f\in C(G)$ with $\hat{f}(\gamma)=0$ for $\gamma\in E$ satisfies $\sum|\hat{f}(\gamma)|<\infty$, or equivalently if for every bounded function $\lambda(\gamma)$ on E, there exists a Baire measure μ on G such that $\hat{\mu}(\gamma)=\lambda(\gamma)$ for $\gamma\in E$ ($\hat{f}$ and $\hat{\mu}$ denote, respectively, the Fourier transform and the Fourier-

Stieltjes transform of f and μ). As measures (operating by convolution) are exactly the operators on $L^1(G)$ which commute with translations, it is natural to define an analogous concept for multipliers of $L^p(G)$, recalling that each $T\in M_p$ corresponds biuniquely to an element $T^\frown$ of $L^\infty(\Gamma)$ (cf. Remark 3).

DEFINITION 2. *A set $E\subseteq\Gamma$ is called a p-Sidon set if every $f\in A_p$ such that $\hat{f}(\gamma)=0$ for $\gamma\notin E$ satisfies $\sum|\hat{f}(\gamma)|<\infty$.*

THEOREM 7. *Let $p\neq 2$, then the following properties are equivalent for a subset E of Γ:*

(i) *E is a p-Sidon set;*

(ii) *if $\lambda\in L^\infty(\Gamma)$, there exists $T\in M_p$ such that $T^\frown(\gamma)=\lambda(\gamma)$ for $\gamma\in E$;*

(iii) *if $\lambda\in L^\infty(\Gamma)$, there exists $T\in M_p$ satisfying* (ii) *and moreover such that $T^\frown(\gamma)=0$ for $\gamma\notin E$;*

(iv) *if $f\in L^1(G)$ and $\hat{f}(\gamma)=0$ for $\gamma\notin E$, then $f\in L^r(G)$ where $r=\max(p, q)$.*

It should be noted that condition (iv) above is the defining property for what is called a *lacunary set of order r* or a $\Lambda(r)$ set. Properties of these sets are investigated in [6] and [4, Chapter VIII]. In particular, it is known that a Sidon set is a lacunary set of order r for every r and hence a p-Sidon set for every p. One should also notice that condition (iii) above implies that the characteristic function of a p-Sidon set is always the transform of a multiplier. The analogous statement for Sidon sets does not hold; indeed it is known that a Sidon set whose characteristic function is the Fourier-Stieltjes transform of a measure is necessarily finite.

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