Lie groups or group representations will be much interested to observe here substantial applications of these theories and indeed some new contributions to these theories. The book is to be well recommended to many mathematicians on many counts but it is an absolute requirement as a source of inspiration to "live-wire topological dynamos" and "spark-plug measurable transformers."

WALTER H. GOTTSCHALK

Elementary methods in analytic number theory.[1] By A. E. Gelfond and U. V. Linnik. Moscow, 1964.

For many years, the theory of numbers has been expanding and growing by leaps and bounds. A great many interesting and significant results have been found, and important developments have been taking place. In due course, brief accounts of these are to be found in the mathematical reviews published in the U.S.A., Germany and the U.S.S.R., and later in encyclopedia articles. However, it has become increasingly difficult for any one to keep in close touch with all that is being done and published in various scattered journals and books.

There are three principal ways in which the new results become more accessible to the reader for closer study. For one, a body of knowledge dealing with closely related and connected results arises and these may be embodied in a systematic treatise. Thus there are Cassel's book on the Geometry of Numbers, the books on Transcendental Numbers by Siegel, Gelfond, and Schneider, Prachars' book on Prime Numbers, Walfisz' book on Exponential Sums.

Next there are books dealing with more loosely connected topics such as Delone and Faddeev's book on *Irrationalities of the third degree*, Vinogradov's book on the *Method of trigonometric sums in the theory of numbers*, and Lang's *Diophantine geometry.*

Finally, the book may contain a collection of miscellaneous topics which are mostly unrelated to each other, but which for various reasons make a special appeal to the author. Khintchine's *Three Pearls* is an instance of this.

The authors of the present book have followed the last pattern and in the twelve chapters of the book have assembled a collection of results representing many different and important aspects of number theory. Some of these results are well known but others have not appeared as yet in treatises. The reader will find a real treasure trove in this book.

Let us glance at the table of contents:

Chapter 1—Additive properties of numbers, the method of Schnirel-

[1] Rand, McNally and Co. are preparing an English translation.

man, and the theorems of Mann and Erdös.

Chapter 2—Elementary solutions of the problems of Waring and of Hilbert-Kamke.

Chapter 3—Problems on the distribution of prime numbers.

Chapter 4—Elementary proofs for the distribution of Gaussian primes and of almost prime Gaussian numbers.

Chapter 5—The sieve of Eratosthenes and Viggo Brun.

Chapter 6—The method of A. Selberg.

Chapter 7—The distribution of the fractional parts of sequences of numbers.

Chapter 8—The number of lattice points in a region.

Chapter 9—The distribution of power residues.

Chapter 10—Elementary demonstration of Hasse's theorem on cubic congruences.

Chapter 11—Elementary demonstration of Siegel's class number theorem for binary quadratic forms.

Chapter 12—The transcendence of some classes of numbers.

The book has a composite authorship. Gelfond wrote Chapters 3, 12; Linnik, Chapters 1, 2, 4, 7, 8, 9, 11; A. I. Vinogradov, Chapters 5, 6, and Manin, Chapter 10.

The term "elementary methods" requires some elucidation. Probably the term had most significance when it applied to prime number theory, e.g., to finding asymptotic formulae without using complex variable theory. It was thought for a long time that this could not be done, but this was disproved by the work of Selberg and Erdös. Chapter 3 contains 40 pages devoted to many aspects of prime number theory. It contains numerous results, e.g., a simple proof that the L series $L(\chi)=\sum_1^\infty \chi(n)/n\neq 0$ when $\chi(n)$ is a non-principal character. This depends on a study of the behavior of the series $\sum_1^\infty \chi(n)x^n/(1-x^n)$. An account is given of the methods of Selberg and Erdös. Much of the chapter, however, involves formidable and not obvious calculations and so some of the proofs are not easily comprehended. The chapter suggests that, sometimes, much is to be said in favor of the non-elementary proofs, the more so since motivation is more obvious.

Similar remarks apply to Chapter 4 on the Gaussian primes.

There are other uses of the term "elementary." It may be applied to proofs free from limiting processes such as infinite series. An instance when an elementary proof is comparatively simple and easily grasped occurs in Chapter 2 on the solutions of Waring's problem and the Hilbert-Kamke problem on the existence of non-negative integer solutions of the n simultaneous equations

$$x_1^r + x_2^r + \cdots + x_s^r = N_r \qquad (r = 1, 2, \cdots, n).$$

Another instance is in Chapter 12 on some classes of transcendental numbers. This deals with the transcendence of e^ω for algebraic $\omega \neq 0$, and of a^b where $a \neq 0$, 1 and b are algebraic numbers where b is not rational, and a, b, ω are all real. Both results depend on the same simple ideas.

These results suggest the desirability in the presentations of mathematical proofs of an introductory account making clear what is involved in the proof, really of course the motivation. Thus we assume that e^ω and a^b are both algebraic. In both of the proofs, use is made of functions of x which are practically the same in both cases, namely, in the first case $f(x) = \sum_{n,k=0}^{p} C_{k,n} x^k e^{\omega n x}$, and in the second $f(x) = \sum_{n,k=0}^{p} C_{k,n} a^{kx} b^{nx}$ and p is a large integer. We can select for the constants $C_{k,n}$ rational integers such that $d^\nu f(x)/dx^\nu = 0$ for large specified ν and a large set of integer values for x. The mean value theorem in the differential calculus leads to an upper estimate for $|f^{(\nu)}(x)|$. The norm of $f^{(\nu)}(x)$, an algebraic number, is either zero or bounded trivially below and this gives a lower estimate for $|f^{(\nu)}(x)|$ which is shown to contradict the upper estimate for $|f^{(\nu)}(x)|$. Hence $f^{(\nu)}(x) = 0$ for the set of values of x. This shows that the function $f(x)$ has more zeros than is possible for functions of its type.

The term "elementary" may also apply to proofs free from an abstruse theory. Chapter 10 contains Manin's elementary proof of Hasse's theorem that the number N of solutions of the congruence $y^2 \equiv x^3 + ax + b \pmod{p}$, where p is a prime, satisfies $|N - p| < 2\sqrt{p}$. The original proof depended upon the theory of elliptic function fields as developed by Hasse and required considerable study.

Some of the most sought-for results have also been found by developing the simplest elementary ideas, and often no other methods are available. There is Mann's density theorem. Let $a_1 > 0, a_2, \cdots, a_n$, be a sequence of non-decreasing integers and let $A(n) = \sum_{a_i \leq n} 1$, and $d(A) = \lim\inf A(n)/n$. Similarly for $b_1, b_2, \cdots$. Then $d(A+B) \geqq \min(d(A)+d(B), 1)$. The proof depends essentially on a counting process. This also applies to Erdös' theorem on essential components of sequences.

Other instances in which only elementary methods are available are given in Chapter 5, which contains applications of Viggo Brun's sieve method. The proof of the asymptotic formula for the number of twin primes, namely, $\pi_2(x) < cx(\log\log x)^2/\log^2 x$ is very simple. The other application shows that every large even number $2N$ can be represented as $2N = q_1 + q_2$ where neither q_1 nor q_2 has more than seven prime factors. The proof is rather complicated. The result is not as good as Buchstab's four factors.

Chapter 6 gives an application of A. Selberg's extension of the sieve method. This yields a sharper estimate for $2\pi_2(n)$. Then with the help of Schnirelman's density theorem, it is shown that every number is the sum of a bounded number of primes.

The term "analytic number theory" also requires consideration. The table of contents and what has just been said suggest that the term is used rather loosely. Originally it was used to indicate those parts of number theory in which results were found by limiting processes or in the proofs of which analysis was applied.

This is exemplified by Dirichlet's theorems on the existence of an infinity of primes in an arithmetical progression and on the class number of binary quadratic forms with given determinant. Other illustrations are questions on the number of lattice points in a circle or hyperbola. Finally there are the results of Hadamard and De la Vallée Poussin on the prime number theorem.

It can be said in favor of analytic methods that the motivation is often more easily seen, and this may not be so obvious in elementary proofs. An instance is that in Chapter 11 on Siegel's theorem on the binary quadratic class number $h(-D)$, where $-D<0$ is a fundamental discriminant, namely, that $h(-D)>D^{1/2-\epsilon}$. The account given in the book, though very ingenious, is complicated and clouded by numerical details. The final step states that if $\chi_k(n)$ is a primitive character mod $k>2$ and $N\geqq k^{50}$, and if $Q(\beta)=\sum_{m\leqq N}\chi_k(m)/m^\beta$, then for $\beta=1+.01\epsilon$, $2Q(\beta)>C(\epsilon)e^{-C(\log k)^{.08}}$, and for $\beta=1-.01\epsilon$, $2Q(\beta)<-C(\epsilon)e^{-C_1(\log k)^{.08}}$.

There are other density problems besides those in Chapter 1, especially distribution problems. These have usually been dealt with by analytical methods of various kinds. Let $\{\beta_1\}, \{\beta_2\}, \cdots, \{\beta_n\}$ be the fractional parts of the β's. Denote by $N_n(\gamma)$ the number of the $\{\beta\}$ for which $\{\beta_j\}\leqq\gamma$, $j\leqq n$, and write $V_n(\gamma)=N_n(\gamma)/n$. Many results are concerned with the behavior of $V_n(\gamma)$ as $n\to\infty$. If $V_n(\gamma)\to V(\gamma)$ say, then the question is to find an estimate for R_n in $V_n(\gamma)=V(\gamma)+R_n$. Thus $R_n=O(n/\rho(n))$ where sometimes $\rho(n)=n^\alpha$ or $(\log r)^\alpha$. Chapter 7 is devoted to Vinogradov's method applied to $\{ap/q\}$ when p runs through the prime numbers and a and q are given integers. The investigation of sums such as $\sum\{f(x)/p\}$ plays an important part.

Chapter 9 deals with the distribution of power residues mod p. Here are contained some interesting and difficult problems in what one might reasonably expect to be elementary number theory, but one soon finds that they are connected with some very deep parts of the theory. Let $\chi(x)$ be a quadratic character mod p. It has been long known that

$$\sum_{x=N_1}^{N_2} \chi(x) = O(p^{1/2} \log p),$$

where the constant implied in O is independent of N_1, N_2. Sharper forms of this result were long sought in vain. It was only a few years ago that an improvement of this result was found by Burgess. He showed that for arbitrary small constants $\delta > 0$, $\epsilon > 0$, and p, $p > p(\epsilon, \delta)$, $H > p^{1/4+\delta}$, and arbitrary N,

$$\left| \sum_{n=N+1}^{N+H} \left(\frac{n}{p}\right) \right| < \epsilon H.$$

Estimates of character sums have important applications, e.g., to finding estimates for the least primitive root and the least non-quadratic residue S mod p. It has been conjectured that $Sp^{-\epsilon} \to 0$ for arbitrary $\epsilon > 0$ as $p \to \infty$. This is very far from the best known result, due to Burgess, that $S = O(p^{\delta + 1/4\sqrt{e}})$ for arbitrary $\delta > 0$.

Questions on the number of lattice points in a region have been studied for many years and various methods have been used, e.g., arithmetical, geometrical, real variable, complex variable. The fundamental problem is really that of finding an estimate for $\sum_{x=1}^{N} \{f(x)\}$ for large N. In Chapter 8, Vinogradov shows that results can be found by elementary methods. In particular he proves the

THEOREM. *Let k, A, Q, R be real numbers for which $k \geqq 1$, $A > 25$, $Q < R$. Suppose that for $Q \leqq x \leqq R$, $f(x)$ has a second derivative $f''(x)$ where $1/kA \leqq f''(x) \leqq 1/A$. Then*

$$\sum_{x>Q}^{x \leqq R} \{f(x)\} = \frac{R-Q}{2} + E,$$

where

$$|E| \leqq 2k\left(\frac{R-Q}{A} + 1\right)(A \log A)^{2/3}.$$

Problems of this kind often require estimates for exponential sums such as

$$\sum_{x>Q}^{x \leqq R} e^{2\pi i f(x)}.$$

This has led to some of the most important results in number theory.

While many of the proofs are clearly expounded, two criticisms might be made. The treatment is now and then condensed and calculations are sometimes omitted. It is also occasionally assumed that the reader is familiar with results which he reasonably might not know. Thus in the proof of the formula for $\pi_2(n)$, in Chapter 5, reference is made without explanation to "the method of stopping at an even term," and this really means that one assumes that

$$1 - k + \frac{k \cdot k - 1}{2!} + \cdots + \frac{k \cdot k - 1 \cdots k - n + 1}{n!} > 0$$

if n is even. So in Chapter 12, on transcendental numbers, reference is made to a theorem of Polya without any statement of the theorem, which however is given in Vol. 2 of Pólya and Szegö's *Aufgaben und Lehrsätze*, Problem 75, Chapter V.

The other criticism refers to results in which formidable calculations play an important part. They are sometimes set out in such a manner that it is not easy to follow the sequence of operations, and to see how they are changing from line to line.

The details given in this review show that very large cross sections of results in number theory are contained in this book and that diverse methods and proofs are given. The chapters can be read independently of each other and so the reader may browse among them. He is sure to find much of interest to him whatever his tastes are. How fortunate he is in having this opportunity of becoming acquainted with so many exciting aspects of number theory, and this without preliminary study. The authors have done a great service to all interested in number theory, and readers will be very grateful to them.

L. J. MORDELL

Electrodynamics and classical theory of fields and particles. By A. O. Barut. The Macmillan Company, New York, 1964.

Many mathematicians are not aware that recent work in theoretical physics has taken a strongly mathematical turn and has posed problems that would be of professional interest to a wide variety of mathematicians. This book is an excellent place to begin to sample this work: It is an exposition by a physicist of those parts of prequantum field theory that are most important for understanding quantum field theory, and has a clear, geometric-Lie group flavor that the reviewer finds very attractive.