RESEARCH ANNOUNCEMENTS

The purpose of this department is to provide early announcement of significant new results, with some indications of proof. Although ordinarily a research announcement should be a brief summary of a paper to be published in full elsewhere, papers giving complete proofs of results of exceptional interest are also solicited.

FINITE SECTION WIENER-HOPF EQUATIONS ON A COMPACT GROUP WITH ORDERED DUAL[1]

BY I. I. HIRSCHMAN, JR.

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Let Θ be a compact Abelian topological group with dual Ξ on which there is given an order relation "$<$" compatible with the group structure. For $\xi \in \Xi$ and $\theta \in \Theta$ we denote by (ξ, θ) the value of the character ξ at θ. $d\theta$ denotes Haar measure on Θ so normalized that Θ has measure 1. A_0 is the class of those functions $f(\theta)$ of the form

$$f(\theta) = \sum_{\xi} \mathbf{f}(\xi)(\xi, \theta)$$

for which $\|f\|_0$ is finite where

$$\|f\|_0 = \sum_{\xi} | \mathbf{f}(\xi) | .$$

Note that

$$\mathbf{f}(\xi) = \int_{\Theta} f(\theta)(-\xi, \theta) d\theta.$$

DEFINITION 1. A Banach algebra A of complex functions on Θ is said to be of type S if:

1. $A \subset A_0$, and $\|f\|_0 \leqq \|f\|$ for all $f \in A$;
2. $(\xi, \theta) \in A$ for every $\xi \in \Xi$, and finite linear combinations of (ξ, θ)'s are dense in A;
3. $f \in A$, $g \in A_0$ and $|\mathbf{g}(\xi)| \leqq |\mathbf{f}(\xi)|$ for all ξ implies $g \in A$ and $\|g\| \leqq \|f\|$.

Henceforth every algebra A considered will be of type S.

Let us introduce the following families of operators:

$$E^{+}(\eta)f\cdot(\theta) = \sum_{\xi \geqq \eta} \mathbf{f}(\xi)(\xi, \theta); \quad E^{-}(\eta)f\cdot(\theta) = \sum_{\xi \leqq \eta} \mathbf{f}(\xi)(\xi, \theta).$$

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It is apparent, using property 3 of Definition 1, that for all $\eta\in\Xi$ $E^+(\eta)$ and $E^-(\eta)$ are linear operators on A (considered as a Banach space) of norm 1.

Let $c\in A$. We define a linear operator W_c^+ on $E^+(0)A$ by

$$W_c^+f = E^+(0)cf, \qquad f\in E^+(0)A.$$

Similarly

$$W_c^-f = E^-(0)cf, \qquad f\in E^-(0)A.$$

W_c^+ and W_c^- are called the Wiener-Hopf operators associated with c. We shall say that $c\in WH(A)$ if both W_c^+ and W_c^- have bounded inverses. We next introduce finite section Wiener-Hopf operators. For $\eta\geqq 0$ let

$$W_c^+(\eta)f = E^-(\eta)E^+(0)cf, \qquad f\in E^-(\eta)E^+(0)A,$$

and

$$W_c^-(\eta)f = E^+(-\eta)E^-(0)cf, \quad f\in E^+(-\eta)E^-(0)A.$$

$W_c^+(\eta)$ and $W_c^-(\eta)$ are bounded linear operators on the Banach spaces $E^-(\eta)E^+(0)A$ and $E^+(-\eta)E^-(0)A$ respectively.

Our principal result is roughly that if "infinite section" Wiener-Hopf operators W_c^+ and W_c^- both have (bounded) inverses, then so do $W_c^+(\eta)$ and $W_c^-(\eta)$ if η is large enough. Before stating this precisely let us introduce some notation. For $f(\theta)\in A$,

$$f(\theta) = \sum_{\xi} f(\xi)(\xi,\theta),$$

let

$$f^{\#}(\theta) = \sum_{\xi} |f(\xi)|(\xi,\theta).$$

It follows from 3 of Definition 1 that $f^{\#}\in A$, and $\|f^{\#}\| = \|f\|$. We will write

$$f^{\#} \prec g^{\#}$$

if $f, g\in A$ and if

$$|f(\xi)| \leqq |g(\xi)| \qquad \text{for all } \xi\in\Xi.$$

THEOREM 2. *Let $c(\theta)\in WH(A)$. Then there exists $\eta_+\geqq 0$ in Ξ, and $C_+ = C_+^{\#}$ in A, such that if $\eta\geqq\eta_+$ and if $f\in E^-(\eta)E^+(0)A$ then*

a. $$f^{\#} \prec [W_c^+(\eta)f]^{\#}C_+^{\#}$$

and if $\eta \geqq \eta_+$ *the range of* $W_c^+(\eta)$ *is* $E^-(\eta)E^+(0)A$. *This implies in particular that* $W_c^+(\eta)^{-1}$ *exists and*

b. $$\|W_c^+(\eta)^{-1}\| \leqq \|C_+^\#\|.$$

There is a similar result associated with $W_c^-(\eta)$.

Conclusion b. of Theorem 2 was proved by Baxter in [3] for the special case when Θ is the real numbers modulo 1, and Ξ is the additive group of integers, and when the algebra A is of Beurling-Gelfand type. Using his result Baxter obtained a very detailed and precise theory of Szegö polynomials on T. See [1] and [2], and also [4]. Using Theorem 2 we can extend this theory to the groups and algebras described above.

In conclusion I would like to express my indebtedness to and my admiration of Professor Baxter's work.

References

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Washington University