ESSENTIAL BOUNDARY POINTS

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Let $B(D)$ denote the class of all bounded holomorphic functions on the connected open subset D of the Riemann sphere, and call a boundary point z_0 of D *removable* if every $f \in B(D)$ has a holomorphic extension to an open set which contains D and z_0. Boundary points which are not removable are called *essential.* It is known [4] that if z_0 is an essential boundary point of D, then there exists $f \in B(D)$, with $\|f\|_D = 1$, whose cluster set at z_0 is the entire closed unit disc. (The symbol $\|f\|_S$ denotes the supremum of the numbers $|f(z)|$ as z ranges over the set S.) Thus, from one standpoint at least, it appears that every essential boundary point z_0 of D has associated with it some $f \in B(D)$ whose singularity at z_0 is as bad as a singularity can be at any boundary point.

Nevertheless, there are situations in which a set of essential boundary points has many of the properties that are usually associated with interior points. The following construction illustrates this.

Let E be a nonempty compact subset of the real axis R, subject to only one condition: we require that $m(E) = 0$, where m denotes one-dimensional Lebesgue measure. Let $\lambda_0, \lambda_1, \lambda_2, \cdots$ be positive numbers such that $\lambda_0 < 1$, $\lambda_k \to \infty$, and

$$\sum_{k=1}^{\infty} (k\lambda_k)^{-1} = \infty. \tag{1}$$

Let $z_n = x_n + iy_n$ $(n = 1, 2, 3, \cdots)$ be points in the open upper half-plane, located so that the set of all limit points of $\{z_n\}$ is precisely E, and put

$$\alpha_n = \inf\{\lambda_0, y_n\lambda_1, (y_n\lambda_2)^2, (y_n\lambda_3)^3, \cdots\}. \tag{2}$$

Since $\lambda_k \to \infty$, we have $\alpha_n > 0$, and we can therefore choose r_n so that

$$0 < r_n < 2^{-n} y_n \alpha_n \tag{3}$$

and so that the closed circular discs Δ_n with radius r_n and center at $z_n + ir_n$ are disjoint; (2) and (3) imply that

$$\sum_{n=1}^{\infty} y_n^{-k-1} r_n \leqq \begin{cases} \lambda_0 & \text{if } k = 0, \\ \lambda_k^k & \text{if } k = 1, 2, 3, \cdots. \end{cases} \tag{4}$$

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Finally, let D be the complement of

$$E \cup \Delta_1 \cup \Delta_2 \cup \Delta_3 \cup \cdots, \tag{5}$$

and let Γ_n be the boundary of Δ_n. Note that $\infty \in D$.

THEOREM I. *If D is constructed as above, then D has the following properties*:

(i) *Every point of E is an essential boundary point of D.*

(ii) *Every $f \in B(D)$ can be extended to $D \cup E$ by the Cauchy formula*

$$f(z) = f(\infty) + \sum_{n=1}^{\infty} \frac{1}{2\pi i} \int_{\Gamma_n} \frac{f(w)}{w - z}\, dw, \tag{6}$$

and the derivatives $f^{(k)}$ can be extended to $D \cup E$ by

$$f^{(k)}(z) = \sum_{n=1}^{\infty} \frac{k!}{2\pi i} \int_{\Gamma_n} \frac{f(w)}{(w - z)^{k+1}}\, dw. \tag{7}$$

(iii) *The series* (6) *and* (7) *converge absolutely and uniformly in the closed lower half-plane G, and the inequalities*

$$\|f^{(k)}\|_G \leq k!\lambda_k^k \|f\|_D \qquad (k = 1, 2, 3, \cdots) \tag{8}$$

hold. In particular, $f, f', f'', \cdots$ are uniformly continuous on G.

(iv) *If $f \in B(D)$ and if $\|f\|_E = \|f\|_D$, then f is constant.*

(v) *If $f(x) = 0$ for infinitely many $x \in E$, then $f(z) = 0$ for all $z \in D$.*

The proofs are quite straightforward. Since every neighborhood of every point of E contains some Δ_n, we have (i). Since every $f \in B(D)$ has nontangential boundary values almost everywhere on each Γ_n, and since $m(E) = 0$, it is easy to see that (6) and (7) hold for all $z \in D$. If $z \in G$, the absolute value of the nth summand in (6) is no larger than $y_n^{-1} r_n \|f\|_D$, the absolute value of the nth summand in (7) does not exceed

$$k! y_n^{-k-1} r_n \|f\|_D, \tag{9}$$

and hence (ii) and (iii) follow from (4).

In particular, we have

$$\|f\|_E \leq \lambda_0 \|f\|_D \tag{10}$$

if $f(\infty) = 0$. If $\|f\|_D = 1$, if f is not constant, and if $f(\infty) = \alpha$, we can apply (10) to the function

$$g = \frac{f - \alpha}{1 - \bar{\alpha} f} \tag{11}$$

and conclude that

$$\|f\|_E \leqq \frac{\lambda_0 + |\alpha|}{1 + |\alpha|\lambda_0} < 1. \tag{12}$$

This gives (iv).

The inequalities (8), combined with our assumption (1), imply that the restriction of every $f \in B(D)$ to the real axis R lies in a quasi-analytic class [**3**]. If $x_0 \in E$ is a limit point of real zeros of f, then $f^{(k)}(x_0) = 0$ for $k = 0, 1, 2, \cdots$ (by repeated application of Rolle's theorem to the real and imaginary parts of f on R). The quasi-analyticity of f therefore shows that f vanishes on R, and hence on D, which proves (v).

Thus E acts like a set of interior points as far as the Cauchy formula, the uniqueness theorem, and the maximum modulus theorem are concerned.

Let us now consider the algebra $A(D)$, consisting of all uniformly continuous holomorphic functions on D, which is a Banach algebra relative to the norm $\|f\|_D$, whose maximal ideal space is the closure $\overline{D}$ of D in the Riemann sphere [**1**], and whose Silov boundary is

$$\partial D = E \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \cdots. \tag{13}$$

If E is countable, for instance, and if D is as in Theorem I, (v) shows that we obtain an example of a sup-norm algebra in which each function is determined by its values on a very small subset of the Silov boundary.

Finally, consider the so-called β-topology on the algebra $B(D)$. This was introduced by Buck [**2**] and is also called the "strict" topology; a typical β-neighborhood of a function $f \in B(D)$ is determined by a continuous real function ϕ on $\overline{D}$, positive on D and 0 on ∂D, and it consists of all $g \in B(D)$ for which

$$\|(g - f)\phi\|_D < 1. \tag{14}$$

If D is the unit disc, Buck has proved (unpublished) that the only β-continuous complex homomorphisms of $B(D)$ are the evaluations at points of D. Rubel and Shields (in a paper which is in preparation) have recently extended this to any D whose boundary has no component consisting of a single point. This result cannot be extended to every D, however, even if every boundary point of D is essential:

THEOREM II. *If D is as in Theorem* I, *if $x \in E$, and if $\Phi(f) = f(x)$, then Φ is a β-continuous homomorphism of $B(D)$.*

It is clear that Φ is a homomorphism, and the Cauchy formula (6), with $z=x$ (and with the curves Γ_n replaced by nearby curves in D), shows that $\Phi(f)$ is obtained by integrating f with respect to a finite measure in D. This shows that Φ is β-continuous [**2**, p. 99].

References

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