AN OBSTRUCTION TO FINITENESS OF CW-COMPLEXES

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A cell structure is a convenient means of describing a space; thus it is important to reduce such a structure to a simpler one when possible. For example, it remains unsolved whether a compact topological manifold (or more generally, ANR) has the homotopy type of a finite CW-complex. According to Milnor [2], this would follow from the conjecture that any CW-complex which is dominated by a finite complex has the homotopy type of a finite complex, but we show below that this is false.

Let X be a connected CW-complex, with universal cover $\tilde{X}$, and fundamental group π with (integral) group ring Λ. Consider the following conditions:

(i) X is dominated by a complex of finite type (i.e., one with a finite number of cells of each dimension),

(ii) π and all $H_i(\tilde{X})$ are countable,

$(\text{iii})_N$ For $N<i$, $H_i(\tilde{X})=0$ and $H^i(X;\mathcal{B})=0$ for all coefficient bundles $\mathcal{B}$ (in the sense of Steenrod; generalised to non-abelian coefficients if $i=2$).

Our results are as follows:

(A) If (i) holds, X is homotopy equivalent to a complex of finite type.

(B) If Λ is noetherian, (i) is equivalent to: π is finitely presented, and all $H_i(\tilde{X})$ are finitely generated Λ-modules.

(C) If X is dominated by a countable complex, it is homotopy equivalent to one; this condition is equivalent to (ii).

(E) If $(\text{iii})_N$ holds, and $N\neq 2$, X has the homotopy type of an N-dimensional complex, countable if (ii) holds.

(F) X is dominated by a finite complex if and only if (i) and some $(\text{iii})_N$ hold. When this is the case, and $N\geqq 2$, there is an obstruction $\theta(X)$ in the projective class group $\tilde{K}^0(\Lambda)$, which depends only on the homotopy type of X, and is zero for X finite. If $\theta(X)=0$, X has the homotopy type of a finite complex of dimension $\max(3, N)$. For $N\geqq 2$, any finite complex K of dimension N, and $\alpha\in\tilde{K}^0(\pi_1(K))$, there is a complex X, with the $(N-1)$-type of K, satisfying (i) and $(\text{iii})_N$, and with $\theta(X)=\alpha$.

The proofs are mostly by induction; we obtain complexes K^r and r-connected maps $\phi: K\to X$, where K is finite in (A), countable in (C). We then prove that $\pi_{r+1}(\phi)$ is finitely generated (over Λ) in (A), and countable in (C), and that we can always use a set of Λ-generators ($r\geqq 2$) of $\pi_{r+1}(\phi)$ to attach $(r+1)$-cells to K, and extend ϕ over them, to obtain an $(r+1)$-connected map. If X satisfies $(\mathrm{iii})_N$, and $r=N-1$, then $\pi_N(\phi)$ is a projective Λ-module; when it is free, the process above gives a homotopy equivalence.

The crucial step in the proof of (A), which is used again in (F) in showing that $\theta(X)$ is well defined, is the following lemma of Whitehead [5]:

Let P be a finite connected complex, K a connected subcomplex with $\pi_r(P, K)=0$ for $1\leqq r<n$. Then there is a formal deformation (and so homotopy equivalence) $D: P\to Q$ rel K such that for $r<n$, Q has no r-cells outside K, and for $r\geqq n+2$, Q has the same number of r-cells outside K as P does.

We observe that there is an interesting analogy between our obstruction in $\tilde{K}^0(\Lambda)$ (which is the Grothendieck group of finitely generated projective modulo free modules) to existence of finite complexes equivalent to X, and Whitehead's obstruction in $K^1(\Lambda)$ (reduced by $\pm\pi$) to their uniqueness up to formal deformation [5]. We refer the reader to Bass and Schanuel [1] for the relation between $K^0(\Lambda)$ and $K^1(\Lambda)$.

According to Swan [4], $\tilde{K}^0(\Lambda)$ is finite, if π is, and by Rim [3], if π is cyclic of prime order, $\tilde{K}^0(\Lambda)$ is isomorphic to the ideal class group of the corresponding cyclotomic field. This gives several examples both of zero and of nonzero $\tilde{K}^0(\Lambda)$.

The main unsatisfactory feature of the above is our inability to construct 2-dimensional complexes under appropriate hypotheses. Roughly speaking, by the time we have enough 2-cells to give relations between the generators of the fundamental group, we may have too many for the homology.

References

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