THE DIMENSION OF THE SUPPORT OF A RANDOM DISTRIBUTION FUNCTION

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In their paper *Random distribution functions* (Bull. Amer. Math. Soc. 69 (1963), 548–551) L. E. Dubins and D. A. Freedman defined a random distribution function F associated with a probability measure μ on the unit square S whose values are distribution functions on $[0, 1]$. To choose a value F_ω of F they proceed as follows: Points $P(n, j)$ of S are defined inductively for all n and $j=0, \cdots, 2^n$ by setting $P(0, 0)=(0, 0)$, $P(0, 1)=(1, 1)$, $P(n+1, 2j)=P(n, j)$ and $P(n+1, 2j+1)$ equal to the image under the unique affine transformation carrying S onto the rectangle $R(P(n, j), P(n, j+1))$ formed by the vertical and horizontal lines through $P(n, j)$ and $P(n, j+1)$ of a point $P^*(n+1, 2j+1)=(x^*(n, 2j+1), y^*(n, 2j+1))$ chosen according to the distribution μ independently of the previous choices. They showed that $\bigcap_{n=1}^{\infty} \bigcup_{j=0}^{2^n} R(P(n, j), P(n, j+1))$ is the graph of a continuous monotone function $F_\omega(x)$ increasing from 0 to 1 on $[0, 1]$, that is, a distribution function defining a measure $\tilde{F}_\omega(E) = \int_E dF_\omega(x)$ on measurable $E \subset [0, 1]$. The inverse of $F_\omega(x)$ is also a continuous everywhere increasing function which we call $G_\omega(y)$ with corresponding measure $\tilde{G}_\omega(E)$. Let

$$I(n,j) = [x(n,j-1), x(n,j)],$$
$$J(n,j) = [y(n,j-1), y(n,j)]$$

and

$$I(n, x) = I(n,j), J(n, x) = J(n,j) \text{ for that } j \text{ for which } x \in I(n,j).$$

$I(n, y)$ and $J(n, y)$ are defined similarly. Let $I^*(n, 2j+\epsilon) = [0, x^*(n, 2j+1)]$ or $[x^*(n, 2j+1), 1]$ and $J^*(n, 2j+\epsilon) = [0, y^*(n, 2j+1)]$ or $[y^*(n, 2j+1), 1]$ according as ϵ equals 0 or 1. We shall write $|I|$ for the length of the interval I, and $h(a, b)$ for the function on S given by $h(a, b)=a \log b+(1-a) \log_2 (1-b)$. All logarithms are taken to the base 2. For any function $k(x, y)$ on S we set

$$E_\mu(k(x, y)) = \int_0^1 \int_0^1 k(x, y)d\mu(x, y)$$

and

$$\sigma_\mu^2(k(x, y)) = E_\mu([k(x, y) - E_\mu(k(x, y))]^2).$$

THEOREM 1. (a) *If* $\sigma_\mu(h(y, x)) < \infty$ *then*

$$\lim_{n\to\infty} \frac{\log |I(n, x)|}{n} = E_\mu(h(y, x))$$

almost everywhere $(\tilde{F}_\omega)$ *for almost all* ω.

(b) *If* $\sigma_\mu(h(y, y)) < \infty$ *then*

$$\lim_{n\to\infty} \frac{\log |J(n, x)|}{n} = E_\mu(h(y, y))$$

almost everywhere $(\tilde{F}_\omega)$ *for almost all* ω.

(c) *If* $\sigma_\mu(h(x, x)) < \infty$ *then*

$$\lim_{n\to\infty} \frac{\log |I(n, y)|}{n} = E_\mu(h(x, x))$$

almost everywhere $(\tilde{G}_\omega)$ *for almost all* ω.

(d) *If* $\sigma_\mu(h(x, y)) < \infty$ *then*

$$\lim_{n\to\infty} \frac{\log |J(n, y)|}{n} = E_\mu(h(x, y))$$

almost everywhere $(\tilde{G}_\omega)$ *for almost every* ω.

In the proof we will need the following law of large numbers for martingales.

LEMMA. *If* f_n *is* F_n*-measurable, where* F_n *is an increasing sequence of* σ*-fields,* $E(|f_n|) < \infty$, $E(|f_n|^2) = \sigma_n^2$ *with* $\sum_{n=1}^\infty \sigma_n^2/n^2 < \infty$, *and if* $E(f_n | F_{n-1}) = 0$ *for all* n *then* $\lim_{n\to\infty} n^{-1}\sum_{j=1}^n f_j = 0$ *almost everywhere.*

PROOF. $S_n = \sum_{j=1}^n f_j/j$ is a martingale, convergent to some limit Z since $E(S_n^2) \leqq \sum_{j=1}^\infty \sigma_j^2/j^2$ for all n. Hence

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^n f_j = \lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^n j(S_j - S_{j-1}) = \lim_{n\to\infty}\left(S_n - \frac{1}{n}\sum_{j=1}^{n-1} S_j\right)$$
$$= Z - Z = 0$$

with probability one.

PROOF OF THEOREM 1. The proofs of all sections of the theorem are the same so we confine ourselves to the first. Since

$$\frac{1}{n}\log |I(n, x)| = \frac{1}{n}\sum_{k=1}^n \log |I^*(k, x)|$$

the result will follow from the preceding lemma if we can show that $f_k = \log |I^*(k, x)| - E_\mu(h(y, x))$ satisfies $E_Q(f_k | F_{k-1}) = 0$ and $E_Q(f_k^2)$

$=\sigma_\mu^2(h(y, x))$ where F_k is the field generated by the $|I^*(l, x)|$ for $l\leqq k$ and Q is the measure on $[0, 1]x\Omega$ defined by $\int k(x, \omega)dQ = E_\omega(\int_0^1 k(x, \omega)dF_\omega(x))$. Any F_{k-1} measurable function has the form $g(x, \omega)=\sum_{j=1}^{2^{k-1}} g_j x_j(x, \omega)$ where $x_j(x, \omega)$ is 1 or 0 depending on whether x is in $I(k-1, j)$ or not so

$$E_Q(gf_k) = E_\omega\left(\sum_{j=1}^{2^{k-1}} g_j \int_{I(k-1,j)} (\log |I^*(k, u)| - E_\mu(h(y, x)))dF_\omega(u)\right) = 0$$

which shows that $E_Q(f_k|F_{k-1})=0$. The verification that $E_Q(f_k^2) =\sigma_\mu^2(h(y, x))$ is straightforward.

Let $C_\mu=\{I_j\}$ be a set of intervals covering E with $\max_j |I_j| \leqq\mu$. The α-dimensional measure of E is

$$\Gamma_\alpha(E) = \lim_{\mu\to 0} \underset{C_\mu}{\text{g.l.b.}} \sum_{I_j\in C_\mu} |I_j|^\alpha.$$

The Hausdorff-Besicovitch dimension of E is

$$\dim E = \inf(\beta | \Gamma_\beta(E) = 0) = \sup(\beta | \Gamma_\beta(E) = \infty).$$

THEOREM 2. *Under the hypotheses of Theorem* 1, *for almost all* ω, *there exist sets* K_ω, L_ω, *with* $\tilde{F}_\omega(K_\omega)=\tilde{G}_\omega(L_\omega)=1$, *such that for any sets* A *and* B *with* $\tilde{F}_\omega(A)>0$ *and* $\tilde{G}_\omega(B)>0$ *we have*

$$\dim(K_\omega \cap A) = E_\mu\{h(y, y)\}/E_\mu\{h(y, x)\}$$

and

$$\dim(L_\omega \cap B) = E_\mu\{h(x, x)\}/E_\mu\{h(x, y)\}.$$

PROOF. The proofs of the two statements are identical so we will prove only the first. Call the right-hand side of the first equation α. We choose an ω in none of the exceptional sets of the first theorem. Then from the first two conclusions of the first theorem, there is a set K_ω with $\tilde{F}_\omega(K_\omega)=1$, such that $|J(n, x)|=|I(n, x)|^{\alpha+O(1)}$ for all $x\in K_\omega$. For each x in $K_{(\omega)}$ we choose that $I(n, x)$ with smallest n such that $|I(n, x)|<\mu$ and $|J(n, x)|>|I(n, x)|^{\alpha+\epsilon}$. For $x_1, x_2\in I(n, x)$ the choice occurs at the same time so the $I(n, x)$ are disjoint and countable and cover K_ω. Hence

$$1 = \int_{\cup I(n,x)} dF_\omega(x) = \sum_{I(n,x)} |J(n, x)| \geqq \sum |I(n, x)|^{\alpha+\epsilon}$$

so $\Gamma_{\alpha+\epsilon}(K_\omega)\leqq 1$, for every $\epsilon>0$, and hence $\dim K_\omega\leqq\alpha$. Let

$$C(\epsilon_1, \epsilon_2) = [x| \ |J(n, x)| > |I(n, x)|^{\alpha-\epsilon_1} \text{or } |I(n, x)| < 2^{n[E_\mu\{h(y,x)\}-\epsilon_2]}]$$

for infinitely many n. Let $C_\mu(\epsilon_1, \epsilon_2)$ be the union of the intervals $I^*(n, x)$ covering $C(\epsilon_1, \epsilon_2)$ where for $x \in C(\epsilon_1, \epsilon_2)$ n is the smallest n for which the conditions of $C(\epsilon_1, \epsilon_2)$ are satisfied with $|I^*(n, x)| \leqq \mu$. Since $\bigcap_{\mu\to 0} C_\mu(\epsilon_1, \epsilon_2) = C(\epsilon_1, \epsilon_2)$, $\lim_{\mu\to 0} \tilde{F}_\omega(C_\mu(\epsilon_1, \epsilon_2)) = 0$. Suppose $\tilde{F}_\omega(A) = 2s$. Take μ so small that $\tilde{F}_\omega(C_\mu(\epsilon_1, \epsilon_2)) < s$, let $A^* = A \cap c(C_\mu(\epsilon_1, \epsilon_2)) \cap K_\omega$ where c indicates complimentation, and set $M(x) = \tilde{F}(A^* \cap [0, x])$. $M(x)$ is a monotone, continuous function, $M(1) > s$, and $M(x+h) - M(x-h) < (2h)^{\alpha-\epsilon_3}$, where ϵ_3 depends on the choice of ϵ_1 and ϵ_2. This happens since $M(x)$ increases only on $I(n, j)$ which fail to lie in $C_\mu(\epsilon_1, \epsilon_2)$. Hence, if $C_\mu = (I_n)$ is a covering of A^* with $|I_n| < \mu$ then

$$s \leqq \int_{A^*} dM(x) = \sum_n \int_{I_n} dM(x) \leqq \sum |I_n|^{\alpha-\epsilon_3}.$$

Hence $\Gamma_{\alpha-\epsilon_3}(A^*) > s$. By adjusting ϵ_1 and ϵ_2, we can choose any $\epsilon_3 > 0$ so $\dim A^* \geqq \alpha$. Hence, with the previous inequality we have

$$\alpha \leqq \dim A^* \leqq \dim A \cap K_\omega \leqq \dim K_\omega \leqq \alpha.$$

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