

FOURIER SERIES IN SEVERAL VARIABLES

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0. Preface. This article is a survey of certain aspects of the theory of multiple Fourier and trigonometric series. It is by no means meant to be a complete survey; for example, it is practically disjoint with the material covered on the subject in Zygmund's book [38, Chapter 17].

There are eight sections to this survey. §1 is the introduction. §2, §3, and §4 are expository in the sense that the main theorems in each section are proved. §5, §6, and §7 are descriptive. §8 consists of two bibliographies, a bibliography for the survey itself, and a general bibliography.

§2 deals with the now classical theory of the Bochner-Riesz summability of multiple Fourier series and the Abel summability of multiple Fourier series. §3 presents Bochner's counter-example for the critical index in summability theory in considerable detail. §4 is concerned with the uniqueness of multiple trigonometric series and proves the main theorem in the subject so far, i.e. uniqueness under Abel summability (due to the present author). §5 describes some results in conjugate multiple Fourier series defined by means of the Calderón-Zygmund kernel and related topics, i.e. analyticity in several variables. §6 deals with the Riemannian theory of multiple trigonometric series. §7 describes some applications to geometric integration theory and potential theory.

1. Introduction. Operating in k -dimensional Euclidean space, E_k , $k \geq 2$, we shall use the following notation:

$$\begin{aligned}x &= (x_1, \dots, x_k), & y &= (y_1, \dots, y_k), \\ \alpha x + \beta y &= (\alpha x_1 + \beta y_1, \dots, \alpha x_k + \beta y_k), \\ (x, y) &= x_1 y_1 + \dots + x_k y_k, & |x| &= (x, x)^{1/2}, \\ m &= (m_1, \dots, m_k), & (m, x) &= m_1 x_1 + \dots + m_k x_k.\end{aligned}$$

With $f(x)$ a function in L^1 on T_k , the k -dimensional torus

$$\{x; -\pi < x_j \leq \pi, j = 1, \dots, k\}$$

and m an integral lattice point, we shall designate the series

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$$\sum_m f(m) e^{i(m,x)}$$

by $S[f]$ and call it the Fourier series of f where

$$\hat{f}(m) = (2\pi)^{-k} \int_{T_k} e^{-i(m,x)} f(x) dx.$$

It is the purpose of this paper to give a survey of some of the known results concerning $S[f]$ and concerning trigonometric series of the form $\sum_m a_m e^{i(m,x)}$.

With $\Delta = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_k^2$, we observe that $\Delta e^{i(m,x)} = -|m|^2 e^{i(m,x)}$. Consequently from an eigenvalue point of view, the first natural question to ask concerning $S[f]$ is "In what manner, does the series

$$\sum_{n=1}^{\infty} \left(\sum_{|m|^2=n} f(m) e^{i(m,x)} \right)$$

approximate f ?" Bearing in mind the classical counter-examples of Fejer and Lebesgue [37, Chapter 8] concerning the one-dimensional Fourier series of continuous functions, we see that the answer to the above question should be phrased in terms of some summability method. The two most natural methods are those of Bochner-Riesz and Abel. In particular, we shall say that $S[f]$ is Bochner-Riesz summable of order α , henceforth designated by $(B-R, \alpha)$, to $f(x)$ if

$$\lim_{R \rightarrow \infty} \sum_{|m| \leq R} f(m) e^{i(m,x)} (1 - |m|^2/R^2)^\alpha = f(x).$$

Bochner-Riesz summability plays the same role for multiple Fourier series that Cesaro summability plays for one-dimensional Fourier series. For further comments on summability methods for multiple Fourier series, see the classical paper of Bochner [3].

We say $S[f]$ is Abel summable to $f(x)$ if

$$\lim_{t \rightarrow 0^+} \sum_m f(m) e^{i(m,x) - |m|t} = f(x).$$

The reason for calling this method of summability Abel summability is motivated by the fact that the series

$$\sum_m f(m) e^{i(m,x) - |m|t}$$

is harmonic in upper E_{k+1} , i.e. in the variables (x, t) , for $t > 0$.

2. Summability. Given f in L^1 on T_k (and real-valued unless explicitly stated otherwise), we shall say that f is extended by periodic-

ity to E_k if f is defined in all of E_k and f is periodic of period 2π in each variable. Letting $B(x, r)$ represent the open k -ball with center x and radius r and letting $|B(x, r)|$ represent the k -dimensional volume of $B(x, r)$, i.e. $|B(x, r)| = 2\pi^{k/2}r^k/k\Gamma(k/2)$, the first theorem we prove is the following [3, p. 189];

THEOREM 1. *Let f be in L^1 on T_k and extended by periodicity to all of E_k , and let $S[f] = \sum_m \hat{f}(m)e^{i(m,x)}$. Set*

$$\sigma_R^\alpha(f, x) = \sum_{|m| \leq R} \hat{f}(m)e^{i(m,x)}(1 - |m|^2/R^2)^\alpha.$$

Suppose that $|B(x_0, h)|^{-1} \int_{B(x_0, h)} |f(x) - f(x_0)| dx \rightarrow 0$ as $h \rightarrow 0$. Then

$$\lim_{R \rightarrow \infty} \sigma_R^\alpha(f, x_0) = f(x_0) \quad \text{for } \alpha > (k - 1)/2.$$

It is to be noted that this theorem implies that

$$\sigma_R^\alpha(f, x) \rightarrow f(x)$$

almost everywhere.

The standard technique for proving theorems of this nature is to first establish the analogous result for multiple Fourier integrals and then proceed by some form of the well-known Poisson summation formula [37, p. 68] to multiple Fourier series. We shall establish Theorem 1 in precisely this manner.

If g is in L^1 on E_k , we shall designate the Fourier transform of g by \hat{g} and define \hat{g} in a manner analogous to the Fourier coefficient of a function on T_k , i.e. $\hat{g}(u) = (2\pi)^{-k} \int_{E_k} e^{-i(x,u)} g(x) dx$.

The first lemma we prove is the following:

LEMMA 1. *Let g be in L^1 on E_k . Set*

$$\tau_R^\alpha(g, x) = \int_{B(0, R)} e^{i(x,u)} \hat{g}(u) (1 - |u|^2/R^2)^\alpha du.$$

Suppose that $|B(x_0, h)|^{-1} \int_{B(x_0, h)} |g(x) - g(x_0)| dx \rightarrow 0$ as $h \rightarrow 0$. Then

$$\lim_{R \rightarrow \infty} \tau_R^\alpha(g, x_0) = g(x_0) \quad \text{for } \alpha > (k - 1)/2.$$

(For a good introduction to the theory of multiple Fourier integrals, see [7, Chapter 2].)

We first observe from an iteration of well-known 1-dimensional theorems and from Lebesgue's dominated convergence theorem that

the above lemma is clear for the special case $g(x) = e^{-|x-x_0|^2}$. Consequently, with no loss in generality, we can assume from the start that $g(x_0) = 0$.

Next we observe from Fubini's theorem and the definition of \hat{g} that

$$(1) \quad \tau_R^\alpha(g, x) = (2\pi)^{-k} \int_{E_k} g(y) dy \int_{B(0, R)} e^{i(x-y, u)} (1 - |u|^2/R^2)^\alpha du.$$

Now it turns out that we can represent the inner integral on the right side of (1) in terms of Bessel functions. In particular, we observe from [33, p. 60] and many other places that

$$(2) \quad J_\nu(z) = \frac{z^\nu}{2^{\nu-1}\Gamma(\nu+1/2)\Gamma(1/2)} \int_0^{\pi/2} \cos(z \cos \theta) (\sin \theta)^{2\nu} d\theta$$

for $\nu > -\frac{1}{2}$

and

$$(3) \quad J_{\mu+\nu+1}(z) = \frac{z^{\nu+1}}{2^\nu \Gamma(\nu+1)} \int_0^{\pi/2} J_\mu(z \sin \theta) (\sin \theta)^{\mu+1} (\cos \theta)^{2\nu+1} d\theta$$

for $\mu > -1$ and $\nu > -1$.

We shall use (3) in the following form:

$$(4) \quad \int_0^t J_\mu(s) s^{\mu+1} [1 - s^2/t^2]^\nu ds = 2^\nu \Gamma(\nu+1) J_{\mu+\nu+1}(t) t^{\mu+1-\nu} \quad \text{for } t > 0.$$

Next, designating the inner integral in (1) by $(2\pi)^k H_R^\alpha(x-y)$, we see, using spherical coordinates in E_k (see [13, Chapter 11]) and letting ω_{k-1} designate the $(k-1)$ -dimensional volume of the unit $(k-1)$ -sphere, i.e. $\omega_{k-1} = 2\pi^{k/2}/\Gamma(k/2)$ and $\omega_0 = 2$, that by (2)

$$\begin{aligned} (2\pi)^k H_R^\alpha(x) &= \omega_{k-2} \int_0^R (1 - r^2/R^2)^\alpha r^{k-1} \left[\int_0^\pi e^{ir|x|\cos\theta} (\sin\theta)^{k-2} d\theta \right] dr \\ &= 2\omega_{k-2} \int_0^R (1 - r^2/R^2)^{\alpha} r^{k-1} \\ &\quad \cdot \left[\int_0^{\pi/2} \cos(r|x|\cos\theta) (\sin\theta)^{k-2} d\theta \right] dr \\ &= 2^{(k-2)/2} \Gamma[(k-1)/2] \Gamma(1/2) \omega_{k-2} \\ &\quad \cdot \int_0^R \left\{ (1 - r^2/R^2)^{\alpha} r^{k-1} J_{(k-2)/2}(r|x|) (r|x|)^{-(k-2)/2} \right\} dr. \end{aligned}$$

It then follows from (4) that

$$(5) \quad H_R^\alpha(x) = c(k, \alpha) J_{k/2+\alpha}(R|x|) R^{k/2-\alpha} |x|^{-(k/2+\alpha)}$$

where

$$\begin{aligned} c(k, \alpha) &= 2^{k/2+\alpha-1} \Gamma(1/2) \Gamma(\alpha + 1) \Gamma[(k - 1)/2] (2\pi)^{-k} \omega_{k-2} \\ &= 2^\alpha \Gamma(\alpha + 1) / (2\pi)^{k/2}. \end{aligned}$$

Since the inner integral in (1) was designated by $(2\pi)^k H_R^\alpha(x - y)$, we have from (1) and (5) that

$$(6) \quad \begin{aligned} \tau_R^\alpha(g, x) &= \int_{E_k} g(y) H_R^\alpha(x - y) dy \\ &= c(k, \alpha) R^{k/2-\alpha} \int_{E_k} g(y) J_{k/2+\alpha}(R|x - y|) |x - y|^{-(k/2+\alpha)} dy. \end{aligned}$$

With no loss in generality, we can take $x_0 = 0$. Setting $G(r) = \int_{B(0,r)} |g(y)| dy$, we see from (6) that to prove the lemma we have to show that $G(r) = o(r^k)$ as $r \rightarrow 0$ implies that

$$(7) \quad R^{k/2-\alpha} \int_0^\infty |J_{k/2+\alpha}(Rs)| |s|^{-(k/2+\alpha)} dG(s) = o(1) \quad \text{as } R \rightarrow \infty.$$

To establish (7), we need two further facts concerning Bessel functions (see [35, p. 199]), namely that there is a constant c_ν such that

$$(8) \quad |J_\nu(s)| = c_\nu s^\nu \quad \text{as } s \rightarrow 0 \quad \text{for } \nu > -1,$$

and that

$$(9) \quad |J_\nu(s)| \leq c_\nu s^{-1/2} \quad \text{as } s \rightarrow \infty \quad \text{for } \nu \geq -\frac{1}{2}.$$

Using (8) and the fact that $G(s) = o(s^k)$ as $s \rightarrow 0$, we see that

$$(10) \quad \begin{aligned} R^{k/2-\alpha} \int_0^{R^{-1}} |J_{k/2+\alpha}(Rs)| |s|^{-(k/2+\alpha)} dG(s) &\leq c_{k/2+\alpha} R^k G(R^{-1}) \\ &= o(1) \quad \text{as } R \rightarrow \infty. \end{aligned}$$

Using (9), the fact that g is in L^1 on E_k , and also that $\alpha > (k - 1)/2$, we see that for every $\delta > 0$,

$$(11) \quad \begin{aligned} R^{k/2-\alpha} \int_\delta^\infty |J_{k/2+\alpha}(Rs)| |s|^{-(k/2+\alpha)} dG(s) \\ \leq c_{k/2+\alpha} R^{-[\alpha - (k-1)/2]} \int_{E_{k-B(0,\delta)}} |g(y)| / |y|^{\alpha+(k+1)/2} dy \\ = o(1) \quad \text{as } R \rightarrow \infty. \end{aligned}$$

Finally using (9) once again, we see that

$$\begin{aligned}
 (12) \quad R^{k/2-\alpha} \int_{R^{-1}}^{\delta} |J_{k/2+\alpha}(Rs)| |s|^{-(k/2+\alpha)} dG(s) \\
 \leq c_{k/2+\alpha} R^{-[\alpha-(k-1)/2]} \int_{R^{-1}}^{\delta} dG(s) / |s|^{[\alpha+(k+1)/2]}.
 \end{aligned}$$

From (10), (11), and (12) we see that for every $\delta > 0$,

$$\begin{aligned}
 (13) \quad \limsup_{R \rightarrow \infty} R^{k/2-\alpha} \int_0^{\infty} |J_{k/2+\alpha}(Rs)| |s|^{-(k/2+\alpha)} dG(s) \\
 \leq c_{k/2+\alpha} \limsup_{R \rightarrow \infty} R^{-[\alpha-(k-1)/2]} \int_{R^{-1}}^{\delta} dG(s) / |s|^{[\alpha+(k+1)/2]}.
 \end{aligned}$$

But $G(s) = o(s^k)$ as $s \rightarrow 0$ implies that the right side of (13) goes to zero as $\delta \rightarrow 0$, as an integration by parts shows. Consequently (7) is established, and therefore the lemma is established.

LEMMA 2. Let $S(x)$ be the trigonometric polynomial $\sum_{|m| \leq R_1} b_m e^{i(m,x)}$, i.e. $S(x) = \sum_m b_m e^{i(m,x)}$ where $b_m = 0$ for $|m| > R_1$. For $R > 0$, set $\sigma_R^\alpha(S, x) = \sum_{|m| \leq R} b_m e^{i(m,x)} (1 - |m|^2/R^2)^\alpha$. Then for $\alpha > (k-1)/2$,

$$\sigma_R^\alpha(S, x) = c(k, \alpha) R^{k/2-\alpha} \int_{E_k} S(y) J_{k/2+\alpha}(R|x-y|) |x-y|^{-(k/2+\alpha)} dy$$

where $c(k, \alpha)$ is the constant in (5).

To prove Lemma 2, set $\phi(t) = (1-t^2)^\alpha$, $0 \leq t \leq 1$, and $\phi(t) = 0$ for $t \geq 1$. Then since $S(x)$ is a finite linear combination of exponentials, it is clear that the lemma will follow if we can show that for fixed x and every u

$$\begin{aligned}
 (14) \quad e^{i(u,x)} \phi(|u|/R) \\
 = c(k, \alpha) R^{k/2-\alpha} \int_{E_k} e^{i(u,v)} J_{k/2+\alpha}(R|x-y|) |x-y|^{-(k/2+\alpha)} dy.
 \end{aligned}$$

Set $g(u) = e^{i(u,x)} \phi(|u|/R)$. Then $g(u)$ is a continuous function which is furthermore in L^1 on E_k . If $\hat{g}(y)$ is also in L^1 on E_k , it follows from Lemma 1 and the Lebesgue dominated convergence theorem that $g(u) = \int_{E_k} e^{i(u,v)} \hat{g}(y) dy$. By (8) and (9), $J_{k/2+\alpha}(R|x-y|) |x-y|^{-(k/2+\alpha)}$ is in L^1 on E_k . Therefore to establish Lemma 2 we need only show that

$$\hat{g}(y) = c(k, \alpha) R^{k/2-\alpha} J_{k/2+\alpha}(R|x-y|) |x-y|^{-(k/2+\alpha)}.$$

But $\hat{g}(y) = H_R^\alpha(x-y)$ and this last fact follows from (5). Lemma 2 is therefore established.

LEMMA 3. *Let $f(x)$ be in L^1 on T_k and extended by periodicity to all of E_k . Then there exists a sequence of trigonometric polynomials $\{S^n(x)\}_{n=1}^\infty$ such that $\int_{T_k} |S^n(x) - f(x)| dx \rightarrow 0$ as $n \rightarrow \infty$.*

Set

$$K_n(t) = \sum_{j=-n}^n e^{ijt}(1 - |j|/n),$$

(so that $K_n(t)$ is the well-known Fejer kernel), and set $S^n(x) = (2\pi)^{-k} \int_{T_k} K_n(y_1) \cdots K_n(y_k) f(y+x) dy$. Then, for $0 < \delta < \pi$,

$$\begin{aligned} \int_{T_k} |S^n(x) - f(x)| dx &\leq \int_{-\delta}^\delta \cdots \int_{-\delta}^\delta K_n(y_1) \cdots K_n(y_k) dy \\ &\quad \cdot \int_{T_k} |f(x+y) - f(x)| dx + o(1) \end{aligned}$$

as $n \rightarrow \infty$, and Lemma 3 is established.

We are now ready to prove Theorem 1. We first observe from (5), (8), and (9) that there is a constant $N(\alpha, R)$ and an $\eta > 0$ such that for x in E_k , $|H_R^\alpha(x)| \leq N(\alpha, R) [1 + |x|]^{-(k+\eta)}$. Consequently, the series

$$\sum_m H_R^\alpha(x + 2\pi m) = H_R^{*\alpha}(x)$$

is absolutely convergent, and furthermore

$$\lim_{R_1 \rightarrow \infty} \sum_{|m| \leq R_1} H_R^\alpha(x + 2\pi m) = H_R^{*\alpha}(x)$$

uniformly for x in a bounded domain.

Let $\{S^j(x)\}_{j=1}^\infty$ be the sequence of trigonometric polynomials given by Lemma 3. Then by Lemma 2 for x in a bounded domain,

$$\begin{aligned} \sigma_R^\alpha(S^j, x) &= \int_{E_k} S^j(y) H_R^\alpha(x-y) dy \\ &= \lim_{R_1 \rightarrow \infty} \sum_{|m| \leq R_1} \int_{T_k} S^j(y + 2\pi m) H_R^\alpha(x - y - 2\pi m) dy \\ &= \lim_{R_1 \rightarrow \infty} \sum_{|m| \leq R_1} \int_{T_k} S^j(y) H_R^\alpha(y - x + 2\pi m) dy \\ &= \int_{T_k} S^j(y) H_R^{*\alpha}(y - x) dy. \end{aligned}$$

Consequently since $H_R^{*\alpha}(x)$ is a continuous periodic function and since $\sigma_R^\alpha(S^j, x) \rightarrow \sigma_R^\alpha(f, x)$ as $j \rightarrow \infty$, we conclude that

$$\sigma_R^\alpha(f, x) = \int_{T_k} f(y) H_R^{*\alpha}(y - x) dy.$$

Observing that $f(y)(1 + |y - x|)^{-(k+\eta)}$ is in L^1 on E_k for fixed x since $\int_{B(0, R_1+1) - B(0, R_1)} |f(y)| dy = O(R_1^{k-1})$ as $R_1 \rightarrow \infty$, we see that we can reverse the above argument and obtain that

$$(15) \quad \sigma_R^\alpha(f, x) = \int_{E_k} f(y) H_R^\alpha(y - x) dy.$$

To prove the theorem with no loss in generality we can assume $x_0 = 0$ and $f(x_0) = 0$. Then locally the same proof will apply here as applied in the proof of Lemma 1 (as can be seen easily by comparing (6) and (15)). Therefore to complete the proof of this theorem we need only show that for fixed $\delta > 0$,

$$(16) \quad \lim_{R \rightarrow \infty} \int_{E_k - B(0, \delta)} f(y) H_R^\alpha(y) dy = 0.$$

But

$$\begin{aligned} & R^{k/2-\alpha} \int_{E_k - B(0, \delta)} |f(y)| J_{k/2+\alpha}(R|y|) |y|^{-(k/2+\alpha)} dy \\ & \leq c_{k/2+\alpha} R^{((k-1)/2)-\alpha} \int_{E_k - B(0, \delta)} \frac{|f(y)|}{|y|^{\alpha+(k+1)/2}} dy = o(1) \text{ as } R \rightarrow \infty \end{aligned}$$

and the proof of Theorem 1 is complete.

We next prove a theorem concerning the Abel summability of multiple Fourier series which is a k -dimensional analogue of the well-known theorem of Fatou. (For another analogue which is useful, see [26, p. 606].)

THEOREM 2. *Let f be in L^1 on T_k and extended by periodicity to all of E_k . Let $S[f] = \sum_m \hat{f}(m) e^{i(m, x)}$ and for $t > 0$, set*

$$A_t(f, x) = \sum_m \hat{f}(m) e^{i(m, x) - |m|^t t}.$$

Also set

$$\beta^-(x) = \limsup_{h \rightarrow 0} |B(x, h)|^{-1} \int_{B(x, h)} f(y) dy$$

and

$$\beta_-(x) = \liminf_{h \rightarrow 0} |B(x, h)|^{-1} \int_{B(x, h)} f(y) dy.$$

Then

$$\beta_-(x) \leq \liminf_{t \rightarrow 0} A_t(f, x) \leq \limsup_{t \rightarrow 0} A_t(f, x) \leq \beta^-(x)$$

To prove Theorem 2, we proceed in a similar manner to the proof of Theorem 1. First let $g(x)$ be in L^1 an E_k . Then for $t > 0$,

$$\int_{E_k} \hat{g}(y) e^{i(y, x) - |y|t} dy = (2\pi)^{-k} \int_{E_k} g(u) \left[\int_{E_k} e^{i(x-u, y) - |y|t} dy \right] du.$$

By (2) and (4)

$$\begin{aligned} \int_{E_k} e^{i(x, y) - |y|t} dy &= \omega_{k-2} \int_0^\infty e^{-rt} r^{k-1} \left[\int_0^\pi e^{i|x|r \cos \theta} (\sin \theta)^{k-2} d\theta \right] dr \\ &= 2^{(k-2)/2} \Gamma[(k-1)/2] \Gamma(\frac{1}{2}) \omega_{k-2} \\ &\quad \cdot \int_0^\infty e^{-rt} r^{k-1} J_{(k-2)/2}(r|x|) (r|x|)^{-(k-2)/2} dr. \end{aligned}$$

Now, as easily seen, the following formula [35, p. 386] holds:

$$\int_0^\infty e^{-ar} J_\nu(br) r^{\nu+1} dr = \frac{2a(2b)^\nu \Gamma(\nu + 3/2)}{(a^2 + b^2)^{\nu+3/2} \sqrt{\pi}} \quad \text{for } \nu > -1.$$

Setting $b_k = 2^{k-1} \Gamma[(k-1)/2] \Gamma[(k+1)/2] \omega_{k-2} (2\pi)^{-k}$, we conclude that for $t > 0$,

$$\int_{E_k} \hat{g}(y) e^{i(y, x) - |y|t} dy = b_k t \int_{E_k} g(u) [t^2 + |x - u|^2]^{-(k+1)/2} du.$$

Using the same techniques to pass from Fourier integrals to Fourier series that we used in the proof of Theorem 1, we consequently obtain that

$$(17) \quad A_t(f, x) = b_k t \int_{E_k} f(u) [t^2 + |x - u|^2]^{-(k+1)/2} du.$$

To prove Theorem 2, we see first that it is sufficient to establish $\limsup_{t \rightarrow 0} A_t(f, x) \leq \beta^-(x)$. Next, we observe that we need only establish this last fact for the special case $x = 0$. In other words to prove the theorem, it is sufficient to show that

$$(18) \quad \limsup_{t \rightarrow 0} A_t(f, 0) \leq \beta^-(0).$$

If $\beta^-(0) = +\infty$, then (18) is immediate. Two cases then present themselves, either $\beta^-(0)$ is finite or $\beta^-(0) = -\infty$. (18) will be established in both of these cases if we can show

$$(19) \quad \text{if } \beta^-(0) < \gamma, \quad \text{then } \limsup_{t \rightarrow 0} A_t(f, 0) \leq \gamma.$$

We now establish (19) and consequently the theorem.

We set

$$(20) \quad f_r(0) = |B(0, r)|^{-1} \int_{B(0, r)} f(u) du,$$

and by (19) choose $\delta > 0$ such that for $0 < r < \delta$, $f_r(0) < \gamma$. Observing that $f(u)/|u|^{k+1}$ is in L^1 on $E_k - B(0, \delta)$, we then obtain from (17) that

$$\begin{aligned} \limsup_{t \rightarrow 0} A_t(f, 0) &\leq b_k |B(0, 1)| \limsup_{t \rightarrow 0} t \int_0^\delta [t^2 + r^2]^{-(k+1)/2} dr^k f_r(0) \\ &\leq b_k |B(0, 1)| (k+1) \limsup_{t \rightarrow 0} t \int_0^\delta r^{k+1} f_r(0) [t^2 + r^2]^{-(k+3)/2} dr \\ &\leq \gamma b_k |B(0, 1)| (k+1) \int_0^\infty t^{k+1} [1 + t^2]^{-(k+3)/2} dt. \end{aligned}$$

But $b_k |B(0, 1)| (k+1) \int_0^\infty t^{k+1} [1 + t^2]^{-(k+3)/2} dt = 1$, (19) is established, and the theorem is proved.

It is clear that one also could prove results along the lines of non-tangential Abel summability (see [37, p. 101]) but we will not concern ourselves with matters of this nature here.

For other versions of the theorems proved in this section see [6, Chapter 2] and [11, Chapter 4].

3. Critical index. In this section, we shall examine in detail the situation that prevails in Bochner-Riesz summability (Theorem 1) when α equals the critical index $(k-1)/2$. Some rather interesting things occur. In the first place, a result is obtained for functions in L^1 on E_k whose analogue is false for functions in L^1 on T_k where as usual $k \geq 2$. In the second place, the result is true in one-dimension for functions both in L^1 on E_1 and in L^1 on T_1 . (The critical index for $k=1$ is $\alpha=0$ which is the same as ordinary convergence.)

THEOREM 3 [3, p. 186]. *Let g be in L^1 on E_k . Set*

$$\tau_R^{(k-1)/2}(g, x) = \int_{B(0,R)} e^{i(x,u)} \hat{g}(u) [1 - |u|^2/R^2]^{(k-1)/2} du.$$

Suppose there exists a $\delta > 0$ such that $g(x) = 0$ for x in $B(0, \delta)$. Then $\lim_{R \rightarrow \infty} \tau_R^{(k-1)/2}(g, 0) = 0$.

By (6) we observe that

$$(21) \quad \tau_R^{(k-1)/2}(g, 0) = c[k, (k-1)/2] R^{1/2} \int_{E_{k-B(0,\delta)}} g(y) J_{k-1/2}(R|y|) |y|^{-(k-1)/2} dy.$$

On the other hand, by [35, p. 199], there exists a constant d_ν such that for $r > 0$ and $\nu \geq 1/2$,

$$(22) \quad |J_\nu(r) - 2^{1/2}(r\pi)^{-1/2} \cos[r - \pi/4 - \nu\pi/2]| \leq d_\nu r^{-3/2}.$$

We consequently conclude from (21) and (22) that to establish Theorem 3, we need only show that

$$(23) \quad \lim_{R \rightarrow \infty} \int_{E_{k-B(0,\delta)}} g(y) \cos[R|y| - k\pi/2] |y|^{-k} dy = 0.$$

But (23) follows immediately from the Riemann-Lebesgue lemma for functions in L^1 on E_1 and the proof of Theorem 3 is complete.

We next establish the ingenious counter-example of Bochner [3, p. 193] showing that the analogue of Theorem 3 is false for functions in L^1 on T_k .

THEOREM 4. *Given $1 > \delta > 0$, there exists a function f in L^1 on T_k such that $f(x) = 0$ in $B(0, \delta)$ and such that*

$$\limsup_{R \rightarrow \infty} |\sigma_R^{(k-1)/2}(f, 0)| = \infty,$$

where $\sigma_R^{(k-1)/2}(f, x) = \sum_{|m| \leq R} \hat{f}(m) e^{i(m,x)} (1 - |m|^2/R^2)^{(k-1)/2}$.

To prove Theorem 4, we set

$$(24) \quad R^\beta \sum_{|m| \leq R} e^{i(m,x)} (1 - |m|^2/R^2)^{((k-1)/2)+\beta} = \Phi_R^\beta(x), \quad \beta \geq 0,$$

and shall write $\Phi_R^0(x)$ as $\Phi_R(x)$. Then it is easy to show using the standard techniques that Theorem 4 will follow once we establish the following:

there exists an x_0 in $T_k - B(0, \delta)$ and an increasing sequence

$$(25) \quad \{R_j\}_{j=1}^\infty \text{ such that } \lim_{j \rightarrow \infty} |\Phi_{R_j}(x_0)| = \infty.$$

For consider the Banach space \mathfrak{B} consisting of all real-valued functions in L^1 on T_k which vanish almost everywhere in $B(0, \delta)$. Then $\sigma_{R_j}^{(k-1)/2}(f, 0) = (2\pi)^{-k} \int_{T_k} f(x) \Phi_{R_j}(x) dx$ gives rise to a sequence of bounded linear functionals F_j on \mathfrak{B} , i.e. $F_j(f) = \sigma_{R_j}^{(k-1)/2}(f, 0)$. If the conclusion to Theorem 4 is false, then $\sup_j |F_j(f)|$ is finite for each f in \mathfrak{B} . But then by the Banach-Steinhaus theorem, $\sup_j \|F_j\|$ is finite. However

$$\|F_j\| = (2\pi)^{-k} \sup_{x \text{ in } T_k - B(0, \delta)} |\Phi_{R_j}(x)|.$$

Therefore $\|F_j\| \geq (2\pi)^{-k} |\Phi_{R_j}(x_0)|$ with x_0 given by (25), and consequently $\sup_j |\Phi_{R_j}(x_0)|$ is finite. But this is a contradiction to (25); and we conclude that Theorem 4 is valid.

Therefore to establish Theorem 4, we only need to establish (25). We shall show even more (see Lemma 6 and the paragraph preceding it), namely $\limsup_{R \rightarrow \infty} |\Phi_R(x)| = \infty$ except possibly for a set of measure zero in T_k . In order to do this we need a sequence of lemmas the first of which is the following:

LEMMA 4. *Let $f(t)$ be a real bounded measurable function in $0 < t < \infty$ for which*

$$a(s) = \lim_{T \rightarrow \infty} T^{-1} \int_0^T f(t) e^{-ist} dt$$

exists for every s in $0 \leq s < \infty$. Then $a(s)$ is different from zero for at most a countable set of numbers s which we shall denote by $S = \{s_1, s_2, \dots\}$. If the numbers in S are linearly independent with respect to integer coefficients, then

$$|a(s_1)| + |a(s_2)| + \dots < \infty.$$

(We say the numbers in S are linearly independent with respect to integer coefficients if (c_1, \dots, c_n) is a set of integers with $c_1^2 + \dots + c_n^2 \neq 0$, then $\sum_{j=1}^n c_j s_j \neq 0$.)

We first note that indeed $a(s) \neq 0$ for only a countable set of non-negative real numbers. For let r_1, \dots, r_n be a set of distinct non-negative real numbers, then

$$(26) \quad T^{-1} \int_0^T \left[f(t) - \sum_{j=1}^n a(r_j) e^{ir_j t} \right] \left[f(t) - \sum_{j=1}^n \bar{a}(r_j) e^{-ir_j t} \right] dt \geq 0.$$

Letting M be the L^∞ -norm of f on $(0, \infty)$ and observing that for

$\beta \neq 0$, $T^{-1} \int_0^T e^{i\beta t} dt \rightarrow 0$ as $T \rightarrow \infty$, we see from (26) that $\sum_{j=1}^n |a(r_j)|^2 \leq M^2$; consequently the set S of the lemma is at most countable.

(Note that zero is not in S because of linear independence.)

Next for s_j in S , we write $a(s_j) = |a(s_j)| e^{-i\beta_j}$ and form

$$(27) \quad K_q(t) = \prod_{j=1}^q K(s_j t - \beta_j) \geq 0$$

where $K(t) = 1 + (e^{it} + e^{-it})2^{-1} = 1 + \cos t \geq 0$.

Then from (27) we observe that

$$(28) \quad K_q(t) = 1 + 2^{-1} \sum_{j=1}^q [e^{is_j t} e^{-i\beta_j} + e^{-is_j t} e^{i\beta_j}] + R_q(t)$$

where $R_q(t)$ is a finite linear combination of $e^{i(\epsilon_1 s_1 + \dots + \epsilon_q s_q)t}$ with $\epsilon_p = 0$ or ± 1 and at least two ϵ_p are different from zero. Consequently, because of linear independence, $\epsilon_1 s_1 + \dots + \epsilon_q s_q$ is not in S and not equal to zero. Likewise $-(\epsilon_1 s_1 + \dots + \epsilon_q s_q)$ is not in S and also is not equal to zero. Therefore

$$(29) \quad \lim_{T \rightarrow \infty} T^{-1} \int_0^T f(t) R_q(t) dt = 0 \quad \text{and} \quad \lim_{T \rightarrow \infty} T^{-1} \int_0^T R_q(t) dt = 0.$$

We conclude from (28) and (29) that

$$\begin{aligned} \lim_{T \rightarrow \infty} T^{-1} \int_0^T f(t) K_q(t) dt &= 2^{-1} \sum_{j=1}^q [\bar{a}(s_j) e^{-i\beta_j} + a(s_j) e^{i\beta_j}] \\ &= \sum_{j=1}^q |a(s_j)|. \end{aligned}$$

On the other hand by (27), (28), and (29)

$$\lim_{T \rightarrow \infty} T^{-1} \int_0^T f(t) K_q(t) dt \leq M \lim_{T \rightarrow \infty} T^{-1} \int_0^T K_q(t) dt = M.$$

We conclude that

$$\sum_{j=1}^q |a(s_j)| \leq M$$

and Lemma 4 is established.

From Lemma 4, we see that the first step in establishing (25) is to show that $R^{-1} \int_0^R \Phi_r(x) e^{-i\lambda r} dr$ tends to a finite limit for $0 \leq \lambda < \infty$. We do this with the following lemma:

LEMMA 5. *If x is not of the form $2\pi m$ where m is a lattice point, then for each λ in $0 \leq \lambda < \infty$*

$$\lim_{R \rightarrow \infty} R^{-1} \int_0^R \Phi_r(x) e^{-i\lambda r} dr$$

exists. It equals 0 if $\lambda \neq \lambda_m(x)$ for every m , and it equals $c_k |\lambda_m(x)|^{-k}$ if $\lambda = \lambda_m(x)$ where

$$\lambda_m(x) = |2\pi m - x|$$

and c_k is a nonzero constant.

To establish Lemma 5, we first establish the following remark [31, p. 91].

REMARK 1. Let g be a continuous function in L^1 on E_k and let \hat{g} be its Fourier transform. Suppose there exists a constant A and an $\epsilon > 0$ such that $|g(x)| \leq A(1 + |x|)^{-(k+\epsilon)}$ and $|\hat{g}(x)| \leq A(1 + |x|)^{-(k+\epsilon)}$. Then

$$(30) \quad \sum_m g(x - 2\pi m) = \sum_m \hat{g}(m) e^{i(m, x)}.$$

To prove the above remark, we note that both sides of (30) converge absolutely uniformly for x in any bounded domain and represent periodic functions of period 2π in each variable. Furthermore, the m_0 th Fourier coefficient of the right side of (30) is $\hat{g}(m_0)$. The m_0 th Fourier coefficient of the left side of (27) is

$$(2\pi)^{-k} \int_{T_k} e^{-i(m_0, x)} \sum_m g(x - 2\pi m) dx = (2\pi)^{-k} \int_{E_k} g(x) e^{-i(m_0, x)} dx = \hat{g}(m_0)$$

and the remark is established.

Taking $g(x)$ in Remark 1 as $J_{k+\beta-1/2}(R|x|)/|x|^{k+\beta-1/2}$, $\beta > 0$, we observe from (14) with $\alpha = (k-1)/2 + \beta$ and $c(k, \alpha) = 2^\alpha \Gamma(\alpha+1)/(2\pi)^{k/2}$ that

$$\begin{aligned} c(k, (k-1)/2 + \beta) R^{1/2-\beta} \int_{E_k} e^{-i(y, x)} \frac{J_{k+\beta-1/2}(R|x|)}{|x|^{k+\beta-1/2}} dx \\ = (1 - |y|^2/R^2)^{(k-1)/2+\beta} \quad \text{for } |y| \leq R \\ = 0 \quad \text{for } |y| > R. \end{aligned}$$

We conclude from Remark 1 and (24) with

$$b(k, \beta) = c(k, (k-1)/2 + \beta) (2\pi)^{k+1/2}$$

that

$$(31) \quad \Phi_R^\beta(x) = \frac{b(k, \beta) R^{1/2}}{(2\pi)^{1/2}} \sum_m \frac{J_{k+\beta-1/2}[R\lambda_m(x)]}{[\lambda_m(x)]^{k+\beta-1/2}}.$$

We next obtain from [35, p. 197] that there exists a constant $A(k)$ such that for $0 < \beta < 1/4$ and $t > 0$,

$$\left| J_{k+\beta-1/2}(t) - (2/\pi t)^{1/2} \cos [t - (k + \beta)\pi/2] \right| < A(k)/t.$$

Consequently from (31), we see that for $0 < \beta < 1/4$ and $R > 0$,

$$\left| \Phi_R^\beta(x) - [b(k, \beta)/\pi] \sum_m \cos \frac{[R\lambda_m(x) - (k + \beta)\pi/2]}{[\lambda_m(x)]^{k+\beta}} \right| < B(k)/R^{1/2}$$

where $A(k)$ and $B(k)$ are constants depending on k but not on β . (Of course, $B(k)$ also depends on x , which is fixed and not of the form $2\pi m$.)

Therefore for $R > 0$ and $1/4 > \beta > 0$,

$$(32) \quad \left| R^{-1} \int_0^R \Phi_r^\beta(x) e^{-ir\lambda} dr - b(k, \beta)(2\pi R)^{-1} \sum_m \left\{ e^{-i(k+\beta)\pi/2} \int_0^R e^{ir[\lambda_m(x)-\lambda]} dr + e^{i(k+\beta)\pi/2} \int_0^R e^{-ir[\lambda_m(x)+\lambda]} dr \right\} / [\lambda_m(x)]^{k+\beta} \right| < \frac{2B(k)}{R^{1/2}}.$$

If $\lambda \neq \lambda_m(x)$ for every lattice point m , we see from (32) that

$$\left| \frac{1}{R} \int_0^R \Phi_r^\beta(x) e^{-ir\lambda} dr - \frac{b(k, \beta)}{2\pi R} \sum_m \frac{e^{-i(k+\beta)\pi/2} [e^{i(\lambda_m(x)-\lambda)R} - 1]}{i[\lambda_m(x) - \lambda][\lambda_m(x)]^{k+\beta}} + \frac{b(k, \beta)}{2\pi R} \sum_m \frac{e^{i(k+\beta)\pi/2} [e^{-i(\lambda_m(x)+\lambda)R} - 1]}{i[\lambda_m(x) + \lambda][\lambda_m(x)]^{k+\beta}} \right| < \frac{2B(k)}{R^{1/2}}.$$

Passing to the limit as $\beta \rightarrow 0$, we see that for $R > 0$,

$$\left| R^{-1} \int_0^R \Phi_r(x) e^{-ir\lambda} dr - \frac{b(k, 0)}{2\pi R} \sum_m e^{-ik\pi/2} \frac{[e^{i[\lambda_m(x)-\lambda]R} - 1]}{i[\lambda_m(x) - \lambda][\lambda_m(x)]^k} + \frac{b(k, 0)}{2\pi R} \sum_m e^{ik\pi/2} \frac{[e^{-i[\lambda_m(x)+\lambda]R} - 1]}{i[\lambda_m(x) + \lambda][\lambda_m(x)]^k} \right| \leq \frac{2B(k)}{R^{1/2}}.$$

Consequently if $\lambda \neq \lambda_m(x)$ for every m ,

$$\lim_{R \rightarrow \infty} R^{-1} \int_0^R \Phi_r(x) e^{-ir\lambda} dr = 0.$$

On the other hand proceeding as above we see from (32) that if $\lambda = \lambda_{m_0}(x)$,

$$\lim_{R \rightarrow \infty} R^{-1} \int_0^R \Phi_r(x) e^{-i\lambda r} dr = \frac{b(k, 0) e^{-ik\pi/2}}{2\pi [\lambda_{m_0}(x)]^k}.$$

Since

$$\begin{aligned} \frac{b(k, 0)}{2\pi} &= (2\pi)^{k-1/2} c(k, (k-1)/2) \\ &= \frac{(2\pi)^{k-1/2}}{(2\pi)^{k/2}} 2^{(k-1)/2} \Gamma(k+1/2), \end{aligned}$$

we see that the lemma is established.

Since every x in $T_k - 0$ is not of the form $2\pi m$ and since furthermore $\sum_m |\lambda_m(x)|^{-k} = +\infty$, we see from Lemmas 4 and 5 that we cannot simultaneously have $\sup_{0 < R < \infty} |\Phi_R(x)| < \infty$ and $\{\lambda_m(x)\}_m$ linearly independent with respect to integer coefficients. Consequently to establish (25) and therefore the theorem, it is sufficient to establish the following lemma ([3, p. 190] and [30, p. 166]):

LEMMA 6. *The sequence $\{\lambda_m(x)\}_m$ is linearly independent with respect to integer coefficients for almost every x in T_k .*

Let $\{m^j\}_{j=1}^\infty$ be an enumeration of the lattice points in E_k . We first observe that $\lambda_{m^j}(x)$ is a real-analytic function of x in $E_k - 2\pi m^j$, and consequently for integers c_1, \dots, c_n with $c_1^2 + \dots + c_n^2 \neq 0$, $\sum_{j=1}^n c_j \lambda_{m^j}(x)$ is a real-analytic function in E_k except at the distinct points $2\pi m^1, \dots, 2\pi m^n$. Let $A_{c_1 \dots c_n}$ be the set of points x in E_k such that $\sum_{j=1}^n c_j \lambda_{m^j}(x) = 0$. If $A_{c_1 \dots c_n}$ is a set of positive measure, then it is not difficult to show that

$$(33) \quad \sum_{j=1}^n c_j \lambda_{m^j}(x) = 0 \quad \text{for all } x \text{ in } E_k.$$

But (33) is impossible if $c_1^2 + \dots + c_n^2 \neq 0$. For suppose $c_{j_1} \neq 0$. Then (33) implies that $\partial \lambda_{m^{j_1}}(x) / \partial x_1$ tends to a finite value as $x \rightarrow 2\pi m^{j_1}$. On the other hand an easy computation shows that

$$\lim_{x \rightarrow 2\pi m^{j_1}} \partial \lambda_{m^{j_1}}(x) / \partial x_1$$

does not exist. We conclude that $A_{c_1 \dots c_n}$ is of measure zero in E_k .

Let A designate the (countable) union of all such $A_{c_1 \dots c_n}$. Then A is of measure zero in E_k . If x is not in A , $\{\lambda_{m^j}(x)\}_{j=1}^\infty$ is linearly independent with respect to integer coefficients. The lemma, and consequently the theorem, is established.

We close this section by stating some recent theorems of Stein concerning Bochner-Riesz summability of multiple Fourier series at the critical index $\alpha = (k-1)/2$.

Let $C(x, r)$ be the $(k-1)$ -sphere which is the boundary of the k -ball $B(x, r)$, and, as before, let ω_{k-1} be the $(k-1)$ -volume of $C(0, 1)$, the unit $(k-1)$ -sphere [see the paragraph between (4) and (5)].

Setting $f^r(x) = (\omega_{k-1}r^{k-1})^{-1} \int_{C(x,r)} f(y) dc(y)$ where $dc(y)$ is the natural $(k-1)$ -volume element on $C(x, r)$, and setting $\delta(x, r) = f^r(x) - f(x)$, we shall say f satisfies a Dini condition at x if there exists an $\eta > 0$ such that

$$\int_0^\eta r^{-1} |\delta(x, r)| dr < \infty.$$

The following theorem then prevails [31, p. 107],

THEOREM 5. *Let $|f| \log^+ |f|$ be in L^1 on T_k and let f be extended by periodicity to all of E_k . Also, let $S[f] = \sum_m \hat{f}(m) e^{i(m,x)}$. Suppose that f satisfies a Dini condition at x_0 . Then*

$$\lim_{R \rightarrow \infty} \sum_{|m| \leq R} \hat{f}(m) e^{i(m,x_0)} (1 - |m|^2/R^2)^{(k-1)/2} = f(x_0).$$

We note that if f is in L^p on T_k for $p > 1$, then $|f| \log^+ |f|$ is in L^1 on T_k and that the case f is in L^2 on T_k is already contained in the original paper of Bochner [3, p. 207].

For the proof of the above theorem, we refer the reader to Stein's paper [31]. The essential idea in the proof is to make a nontrivial estimate of the difference of the corresponding "Dirichlet" kernels on T_k and E_k , i.e. of the difference

$$\sum_{|m| \leq R} e^{i(m,x)} (1 - |m|^2/R^2)^{(k-1)/2} - \int_{B(0,R)} e^{i(x,y)} (1 - |y|^2/R^2)^{(k-1)/2} dy.$$

Stein furthermore puts the critical index in proper perspective by establishing the following generalization of Kolmogorov's classical result on one dimensional Fourier series.

THEOREM 6. *There exists an f in L^1 on T_k with $S[f] = \sum_m \hat{f}(m) e^{i(m,x)}$ such that for almost every x in T_k ,*

$$\limsup_{R \rightarrow \infty} \left| \sum_{|m| \leq R} \hat{f}(m) e^{i(m,x)} (1 - |m|^2/R^2)^{(k-1)/2} \right| = +\infty.$$

For the proof of the above theorem, we refer the reader to Stein's paper [30, p. 165]. The essential idea in the proof is to apply a theorem proved in the paper concerning weak type operators.

4. Uniqueness. We now turn our attention to a different aspect of multiple Fourier series, namely the uniqueness theory of multiple trigonometric series.

In view of Theorem 2, one is led naturally to the following question:

If $\lim_{t \rightarrow 0+} \sum_m a_m e^{i(m,x)-|m|t} = 0$ for all x in T_k , is $a_m = 0$ for all m ?

Since the series with coefficients a_m defined by $a_m = 0$ for $m_2^2 + \cdots + m_k^2 \neq 0$ and $a_m = m_1$ for $m_2^2 + \cdots + m_k^2 = 0$ has the property that $\lim_{t \rightarrow 0+} \sum_m a_m e^{i(m,x)-|m|t} = 0$ for all x , we see that another condition is needed on the a_m in order to answer the above question in the affirmative. A natural condition, in view of the above, is $\sum_{R-1 < |m| \leq R} |a_m| = o(R)$ as $R \rightarrow \infty$. The following theorem established by Shapiro [25], prevails.

THEOREM 7. *Given the multiple trigonometric series $\sum_m a_m e^{i(m,x)}$ where the a_m are arbitrary complex numbers. Suppose that*

- (i) $\sum_{R-1 < |m| \leq R} |a_m| = o(R)$ as $R \rightarrow \infty$,
- (ii) $\lim_{t \rightarrow 0} \sum_m a_m e^{i(m,x)-|m|t} = 0$ for x in $T_k - 0$.

Then $a_m = 0$ for all m .

Before proving the above theorem, we would like to make a number of comments. In the first place, in view of the preceding remarks the condition (i) cannot be replaced by $\sum_{R-1 < |m| \leq R} |a_m| = o(R)$ as $R \rightarrow \infty$. In the second place the above theorem is false in one-dimension (we are assuming throughout this survey that $k \geq 2$) as is easily seen from a consideration of the Fourier-Stieltjes series of the unit mass placed at the origin in T_1 . However, if (ii) is replaced by the condition "0 in all of T_1 ," then the above theorem is true in one-dimension and is due to Verblunsky. (For a history of the aspects of the above theorem, see [37, pp. 382-383] and many other places.) The k -dimensional version requires several new and interesting ideas.

A much more general version of Theorem 7 involving upper and lower limits of the Abel partial sums and functions in L^1 on T_k actually prevails. For the full statement and proof of these theorems due to Shapiro, we refer the reader to [25] (and also to [27, Chapter 1] where analogous theorems for the two-sphere are established. See also [19; 12; 17; 18]).

In order to prove Theorem 7, we first need the concept of an upper and lower generalized Laplacian which we define as follows:

Let $G(x)$ be a function in L^1 in a neighborhood of the point x_0 , then using the notation previously introduced (see (20)), set $G_h(x_0) = |B(x_0, h)|^{-1} \int_{B(x_0, h)} G(x) dx$. Then we call $\Delta^* G(x_0)$ the upper generalized Laplacian of G at x_0 and define it by

$$\Delta^*G(x_0) = 2(k + 2) \limsup_{h \rightarrow 0} [G_h(x_0) - G(x_0)]/h^2.$$

In a similar manner, we define Δ_* using $\lim \inf_{h \rightarrow 0}$.

It is to be noticed that if $G(x)$ is also in class C^2 in a neighborhood of x_0 , then $\Delta^*G(x_0) = \Delta_*G(x_0) = \Delta G(x_0)$ where Δ designates the usual Laplace operator.

The following lemma then prevails:

LEMMA 7. *Let $G(x)$ be a function in L^1 on T_k and extended by periodicity to all of E_k . Let $S[G] = \sum_m \hat{G}(m)e^{i(m,x)}$ and for $t > 0$, set $G(x, t) = \sum_m \hat{G}(m)e^{i(m,x) - |m|t}$. Suppose $\lim_{t \rightarrow 0+} G(x_0, t) = G(x_0)$ exists and is finite. Set $\gamma^*(x_0) = \limsup_{t \rightarrow 0+} - \sum_m |m|^2 \hat{G}(m)e^{i(m,x_0) - |m|t}$ and define $\gamma_*(x_0)$ similarly using $\lim \inf_{t \rightarrow 0+}$. Then*

- (a) $\Delta_*G(x_0) \leq \gamma^*(x_0)$ and
- (b) $\gamma_*(x_0) \leq \Delta^*G(x_0)$.

(For the one-dimensional analogue of the above result see [37, p. 353].)

To prove the lemma it is sufficient to prove (a), for (b) will then follow by considering $-G(x)$. With no loss in generality, we can also assume that $x_0 = 0$ and $G(0) = 0$.

If $\Delta_*G(0) = -\infty$, or if $\gamma^*(0) = +\infty$ (a) is already established, so we can also assume that $\Delta_*G(0) > -\infty$ and $\gamma^*(0) < +\infty$.

Suppose (a) does not hold. Then there exists a constant η such that $\Delta_*G(0) > \eta > \gamma^*(0)$. Since we can find a periodic function $\lambda(x)$ which is in class C^∞ with the properties that $\lambda(0) = 0$ and $\Delta\lambda(0) = \eta$, we can assume that $\eta = 0$. We prove the lemma by showing that $\Delta_*G(0) > 0 > \gamma^*(0)$ leads to a contradiction.

First suppose $\gamma^*(0) < 0$. Then with $G_t(0, t)$ designating $dG(0, t)/dt$ we observe that $\limsup_{t \rightarrow 0} -G_{tt}(0, t) = \gamma^*(0) < 0$. Consequently, $G_{tt}(0, t) > 0$ for t sufficiently small, and therefore for t sufficiently small $G_t(0, t)$ is an increasing function of t , i.e. there exists a $t_0 > 0$ such that $G_t(0, t)$ is an increasing function in the interval $0 < t < t_0$. Also $G(0, t)/t = G_t(0, s)$ where $0 < s < t$ by the mean-value theorem, since $\lim_{t \rightarrow 0} G(0, t) = 0$. Therefore $\limsup_{t \rightarrow 0} d[G(0, t)/t]/dt < 0$ is incompatible with the fact that $G_t(0, t)$ is an increasing function for $0 < t < t_0$. Consequently, if we can show that

$$(34) \quad \text{if } \Delta_*G(0) > 0, \quad \text{then } \liminf_{t \rightarrow 0} -d[G(0, t)/t]/dt > 0,$$

we shall have arrived at a contradiction.

We now establish (34). Observing that $|B(0, r)| G_r(0) = \int_{B(0,r)} G(u) du$ we obtain from (17) that for $t > 0$,

$$\begin{aligned} G(0, t)/t &= A \int_0^\infty [t^2 + r^2]^{-(k+1)/2} dr^k G_r(0) \\ &= A' \int_0^\infty r^{k+1} G_r(0) [t^2 + r^2]^{-(k+3)/2} dr, \end{aligned}$$

where A and A' are positive constants. Since $G_r(0) = O(1)$ as $r \rightarrow \infty$, we conclude that

$$(35) \quad d[-G(0, t)/t]/dt = A'' t \int_0^\infty r^{k+1} G_r(0) [t^2 + r^2]^{-(k+5)/2} dt,$$

where A'' is a positive constant. By the assumption in (34), there exists $h_0 > 0$ such that for $0 < r < \delta$, $G_r(0) > r^2 h_0$. We thus obtain from (35) that

$$\begin{aligned} \liminf_{t \rightarrow 0} d[-G(0, t)/t]/dt &\geq A'' \liminf_{t \rightarrow 0} t \int_0^\delta r^{k+3} h_0 [t^2 + r^2]^{-(k+5)/2} dr \\ &\geq h_0 A'' \int_0^\infty r^{k+3} [1 + r^2]^{-(k+5)/2} dr \\ &> 0; \end{aligned}$$

(34) is established, and the proof of the lemma is complete.

We now prove the theorem. From a consideration of the series $\sum_m (a_m + \bar{a}_{-m}) e^{i(m, x)}$ and $\sum_m i(a_m - \bar{a}_{-m}) e^{i(m, x)}$, we see from the start that it is sufficient to prove the theorem under the additional assumption $\bar{a}_m = a_{-m}$, which we shall henceforth make.

We next set for $t > 0$,

$$\begin{aligned} f(x, t) &= \sum_m a_m e^{i(m, x) - |m|t}, \\ f_1(x, t) &= - \sum_{m \neq 0} a_m |m|^{-1} e^{i(m, x) - |m|t}, \\ (36) \quad F(x, t) &= - \sum_{m \neq 0} a_m |m|^{-2} e^{i(m, x) - |m|t}, \\ f^*(x) &= \limsup_{t \rightarrow 0} f(x, t) \quad \text{and} \quad f_*(x) = \liminf_{t \rightarrow 0} f(x, t), \end{aligned}$$

and observe that

$$(37) \quad \text{for } x \text{ in } T_k - 0, \quad \lim_{t \rightarrow 0} f_1(x, t) \quad \text{and} \quad \lim_{t \rightarrow 0} F(x, t) \text{ exist and are finite.}$$

To see this, fix x in $T_k - 0$. Then by (ii), there exists a constant K depending on x such that $|f(x, t)| \leq K$ for $t > 0$. Then by the mean-value theorem for $0 < t_1 < t_2$, there exists an s such that

$|f_1(x, t_2) - f_1(x, t_1)| = |f(x, s) - a_0|(t_2 - t_1)$ where $0 < t_1 < s < t_2$. Therefore, $f_1(x, t)$ satisfies the Cauchy criterion for convergence and the first part of (37) is established. Repeating this same argument for $F(x, t)$ establishes the second part.

Using (37), we define the periodic function $F(x)$ in E_k for $x \neq 2\pi m$ by $F(x) = \lim_{t \rightarrow 0} F(x, t)$. As is easily seen, (i) implies that $\sum_{m \neq 0} |a_m|^2 / |m|^4 < \infty$. Consequently $F(x)$ is in L^2 on T_k and $S[F] = -\sum_{m \neq 0} a_m e^{i(m,x)} / |m|^2$. Using the notation of (20), (i), and the fact that $|B(x, h)|^{-1} \int_{B(x,h)} e^{i(m,v)} dy = \mu_k e^{i(m,x)} J_{k/2}(|m|h) (|m|h)^{-k/2}$ where μ_k is a constant depending on k but not on m , we conclude that for $h > 0$,

$$(38) \quad F_h(x) = -\mu_k \sum_{m \neq 0} a_m J_{k/2}(|m|h) e^{i(m,x)} / |m|^2 (|m|h)^{k/2}.$$

(Note that it is clear from (38), that $\mu_k^{-1} = \lim_{t \rightarrow 0} J_{k/2}(t) t^{-k/2}$. Also, note that the series in (38) is absolutely convergent.)

We next obtain from (ii), (36) and Lemma 7 that

$$(39) \quad \Delta^* F(x) \geq -a_0 \quad \text{and} \quad \Delta_* F(x) \leq -a_0 \quad \text{for } x \neq 2\pi m.$$

Consequently, it follows from well-known theorems concerning generalized Laplacians [21, p. 14] that if $B(x_0, h_0)$ contains no point of the form $2\pi m$ and if $F(x)$ is continuous in $B(x_0, h_0)$, then (39) implies that $F(x) + a_0|x|^2/2k$ is actually harmonic in $B(x_0, h_0)$.

We now show that given such a $B(x_0, h_0)$, $F(x)$ is actually continuous there, which is the heart of the whole proof. In order to accomplish this, we first need to establish some more facts concerning $F(x)$.

We set $\alpha_1(t) = \sup_{0 < h \leq t} \sup_x \text{ in } T_k |F_h(x) - F(x, h)|$ and shall show that

$$(40) \quad \lim_{t \rightarrow 0} \alpha_1(t) = 0.$$

It follows from the definition of $F(x, h)$ in (36) and $F_h(x)$ in (38) that

$$(41) \quad \sup_{x \text{ in } T_k} |F_h(x) - F(x, h)| \leq \sum_{m \neq 0} |a_m| |e^{-|m|h} - \mu_k J_{k/2}(m|h)| (|m|h)^{-k/2} |m|^{-2}.$$

We split the sum on the right side of (41) into two parts, A_h and B_h with A_h designating the sum over the lattice points m , $1 \leq |m| \leq h^{-1}$, and B_h designating the sum over the lattice points m , $|m| \geq h^{-1}$. To establish (40), it is sufficient to show that $\lim_{h \rightarrow 0} A_h = 0$ and $\lim_{h \rightarrow 0} B_h = 0$.

Observing that there is a constant K such that

$$|e^{-r} - \mu_k J_{k/2}(r)r^{-k/2}| \leq Kr \quad \text{for } 0 \leq r < 1,$$

we see from assumption (i) of the theorem and (41) that

$$A_h \leq Kh \sum_{1 \leq |m| \leq h^{-1}} |a_m| |m|^{-1} = ho(h^{-1}).$$

Therefore, $\lim_{h \rightarrow 0} A_h = 0$.

Using the fact that there exists a constant K such that $|J_{k/2}(r)| \leq Kr^{-1/2}$, we obtain from (41) and from assumption (i) of the theorem that for h small,

$$B_h \leq o(1) \int_{h^{-1}-1}^{\infty} e^{-r} r^{-1} dr + o(h^{-(k+1)/2}) \int_{h^{-1}-1}^{\infty} r^{-(k+3)/2} dr.$$

We conclude that $\lim_{h \rightarrow 0} B_h = 0$, and, consequently, that (40) is established.

The next fact we establish concerning $F(x)$ is the following:

Set $\alpha_2(t) = \sup_{0 < h \leq t} \sup_{|x-p| \leq h, x \text{ and } p \text{ in } E_k} |F_h(x) - F_h(p)|$. Then

$$(42) \quad \lim_{t \rightarrow 0} \alpha_2(t) = 0.$$

By (38),

$$\begin{aligned} & \sup_{|x-p| \leq h, x \text{ and } p \text{ in } E_k} |F_h(x) - F_h(p)| \\ & \leq \sup_{|x-p| \leq h, x \text{ and } p \text{ in } E_k} \mu_k \sum_{1 \leq |m| \leq h^{-1}} \\ (43) \quad & \cdot \{ |a_m| |J_{k/2}(|m|h)| (|m|h)^{-k/2} |m|^{-2} |e^{i(m,x)} - e^{i(m,p)}| \} \\ & + 2\mu_k \sum_{h^{-1} \leq |m|} |a_m| |J_{k/2}(|m|h)| (|m|h)^{-k/2} |m|^{-2} \\ & = A'_h + B'_h. \end{aligned}$$

To establish (42), we have to show that both A'_h and B'_h tend to zero as h goes to zero. That $\lim_{h \rightarrow 0} B'_h = 0$ was already shown when we established (40). For A'_h , we observe that

$$\begin{aligned} A'_h & \leq \mu_k \sum_{1 \leq |m| \leq h^{-1}} |a_m| |J_{k/2}(|m|h)| (|m|h)^{-k/2} |m|^{-2} |m|h \\ & \leq O(1)h \sum_{1 \leq |m| \leq h^{-1}} |a_m| |m|^{-1} \\ & \leq O(1)ho(h^{-1}). \end{aligned}$$

Consequently, $\lim_{h \rightarrow 0} A'_h = 0$ and (42) is established.

Next, using the fact that $f(x, t)$ is continuous for $t > 0$ and periodic in the x variables, we select a sequence $t_1 > t_2 > \dots > t_n \dots \rightarrow 0$ such that

$$(44) \quad \sup_{x \text{ in } \bar{E}_k} \sup_{t_{n+1} \leq t \leq t_n} |f(x, t) - f(x, t_n)| \leq 1.$$

Now, let $B(x_0, h_0)$ be an open k -ball whose closure $\bar{B}(x_0, h_0)$ contains no points of the form $2\pi m$, m an integral lattice point. We propose to show that $F(x) + a_0|x|^2/2k$ is continuous in $B(x_0, h_0)$. By (ii) of the theorem, $\lim_{n \rightarrow \infty} |f(x, t_n)| = 0$ for x in $B(x_0, h_0)$. Consequently if $\bar{B}(y, s) \subset B(x_0, h_0)$, $s > 0$, there exists (by the Baire category theorem, see [37, p. 29 (12.3i)]) $\bar{B}(y', s') \subset \bar{B}(y, s)$, $s' > 0$, and a constant K such that $|f(x, t_n)| \leq K$ for x in $\bar{B}(y', s')$ and $n = 1, 2, \dots$. But, then by (44), $|f(x, t)| \leq K + 1$ for $0 < t \leq t_1$ and x in $\bar{B}(y', s')$. Employing the same technique used in establishing (37), we obtain that $f_1(x, t)$ and $F(x, t)$ converge uniformly as $t \rightarrow 0$ for x in $\bar{B}(y', s')$. Consequently, $F(x) + a_0|x|^2/2k$ is continuous in $\bar{B}(y', s')$, and we conclude that the set of points $Z \subset B(x_0, h_0)$ at which $F(x) + a_0|x|^2/2k$ is not continuous must be nowhere dense.

We next observe that Z contains no isolated points. For if z_0 were an isolated point of Z , then there would exist $s_0 > 0$ such that $F(x) + a_0|x|^2/2k$ would be harmonic in the punctured k -ball $B(z_0, 2s_0) - z_0$. Therefore by (37), (40), (42), and the mean-value theorem for harmonic functions, for x in $B(z_0, h) - z_0$ and $0 < h < s_0$

$$\begin{aligned} & |F(z_0) + a_0|z_0|^2/2k - F(x) - a_0|x|^2/2k| \\ &= \left| F(z_0) + a_0|z_0|^2/2k \right. \\ &\quad \left. - \left\{ F_{|z_0-x|}(x) + a_0 \int_{B(x, |z_0-x|)} |y|^2 dy / |B(x, |z_0-x|)| 2k \right\} \right| \\ &\leq |F(z_0) - F_{|z_0-x|}(x)| \\ &\quad + |a_0| \left| \int_{B(x, |z_0-x|)} (|y|^2 - |z_0|^2) dy \right| / |B(x, |z_0-x|)| 2k \\ &\leq |F(z_0) - F_{|z_0-x|}(x)| + o(1) \\ &\leq |F(z_0) - F(z_0, |z_0-x|)| + |F(z_0, |z_0-x|) - F_{|z_0-x|}(z_0)| \\ &\quad + |F_{|z_0-x|}(z_0) - F_{|z_0-x|}(x)| + o(1) \\ &\leq o(1) + \alpha_1(h) + \alpha_2(h) \\ &\leq o(1). \end{aligned}$$

Consequently, $\lim_{x \rightarrow z_0} F(x) + a_0|x|^2/2k = F(z_0) + a_0|z_0|^2/2k$. We conclude that Z can contain no isolated points.

Next, let \bar{Z} designate the closure of Z . Since $|f(x, t)| \rightarrow 0$ for x in

\bar{Z} and since \bar{Z} is a perfect, we can obtain once again from the Baire category theorem and (44) that if \bar{Z} is a nonempty set that there exists a z_0 in Z and an $s_0 > 0$ with $\bar{B}(z_0, 2s_0) \subset B(x_0, h_0)$ and a constant K such that

$$(45) \quad |f(z, t)| \leq K + 1 \quad \text{for } z \text{ in } \bar{Z}\bar{B}(z_0, 2s_0).$$

We propose to show that $F(x) + a_0|x|^2/2k$ is actually continuous at z_0 , and consequently that Z is the empty set, i.e. $F(x) + a_0|x|^2/2k$ is continuous in $B(x_0, h_0)$.

Employing the same technique used in establishing (37), we see from (45), that $f_1(z, t)$ and $F(z, t)$ converge uniformly as $t \rightarrow 0$ for z in $\bar{Z}\bar{B}(z_0, 2s_0)$, and consequently since this latter set is closed that $F(x)$ is continuous for x restricted to $\bar{Z}\bar{B}(z_0, 2s_0)$. Therefore given an $\epsilon > 0$, choose s_1 such that $0 < s_1 < s_0$ and such that

$$(46) \quad |F(z) - F(z_0)| < \epsilon \quad \text{for } z \text{ in } \bar{Z}\bar{B}(z_0, s_1).$$

Next, using (40), (42), and (45) choose s_2 such that

$$(47) \quad \begin{aligned} \alpha_1(s) < \epsilon \quad \text{and} \quad \alpha_2(s) < \epsilon \quad \text{for } 0 < s < s_2; \\ |a_0|(2|z_0| + 3s_2)s_2 < \epsilon; \\ |F(z, s) - F(z)| < \epsilon \quad \text{for } 0 < s < s_2 \quad \text{and } z \text{ in } \bar{Z}B(z_0, 2s_0); \\ 2s_2 < s_1. \end{aligned}$$

We propose to show that

$$(48) \quad |F(x) - F(z_0)| < 5\epsilon \quad \text{for } x \text{ in } B(z_0, s_2).$$

If x is in $B(z_0, s_2)$ and x is in \bar{Z} , then (48) holds by (46) and (47). We can therefore suppose that x is in $B(z_0, s_2)$ and x is not in \bar{Z} . Let z' be the closest point in \bar{Z} (or one of the closest if more than one exists) to x . Then $|z' - x| = s_3 < s_2$ and $F(y) + a_0|y|^2/2k$ is a harmonic function in $B(x, s_3)$. Therefore,

$$(49) \quad \begin{aligned} |F(x) - F(z_0)| &\leq |F(x) - F(z')| + |F(z') - F(z_0)| \\ &\leq |F_{s_3}(x) - F(z')| + |a_0| \\ &\cdot (2k)^{-1} \left| \int_{B(x, s_3)} [|y|^2 - |x|^2] dy \right| |B(x, s_3)|^{-1} \\ &\quad + |F(z') - F(z_0)| \\ &\leq |F_{s_3}(x) - F(z')| + 2\epsilon, \end{aligned}$$

by (46) and (47).

But once again by (47),

$$\begin{aligned} |F_{s_3}(x) - F(z')| &\leq |F_{s_3}(x) - F_{s_3}(z')| + |F_{s_3}(z') - F(z', s_3)| \\ &\quad + |F(z', s_3) - F(z')| \\ &\leq \alpha_2(s_3) + \alpha_1(s_3) + |F(z', s_3) - F(z')| \leq 3\epsilon. \end{aligned}$$

This last part coupled with (49) gives us (48). We conclude first that F is continuous at z_0 , next that Z is empty, and finally that $F(x) + a_0|x|^2/2k$ is harmonic in $B(x_0, h_0)$.

Since $B(x_0, h_0)$ was any open k -ball in E_k whose closure does not contain a point of the form $2\pi m$, we have that $F(x) + a_0|x|^2/2k$ is harmonic in $E_k - \bigcup_m \{2\pi m\}$.

To complete the proof of the theorem, we need the following lemma.

LEMMA 8. For $t > 0$, set

$$H_0(x, t) = \sum_{m \neq 0} e^{i(m, x) - |m|t} / |m|^2,$$

and for $j = 1, \dots, k$,

$$H_j(x, t) = \sum_{m \neq 0} im_j e^{i(m, x) - |m|t} / |m|^2.$$

Then for $j = 0, 1, \dots, k$,

$\lim_{t \rightarrow 0} H_j(x, t) = H_j(x)$ exists and is finite for $x \neq 2\pi m$, $H_j(x)$ is in L^1 on T_k ,

and

$$\lim_{t \rightarrow 0} \int_{T_k} |H_j(x, t) - H_j(x)| dx = 0.$$

Set

$$\begin{aligned} \Phi(x) &= (2\pi)^k [\omega_{k-1}(k-2)]^{-1} |x|^{-(k-2)} \quad \text{for } x \text{ in } T_k - 0 \quad \text{and } k \geq 3 \\ &= (2\pi) \log |x|^{-1} \quad \text{for } x \text{ in } T_2 - 0 \quad \text{and } k = 2, \end{aligned}$$

and extend $\Phi(x)$ by periodicity to all of E_k . Then

$$\lim_{|x| \rightarrow 0} [H_0(x) - \Phi(x)] \text{ exists and is finite}$$

and

$$H_0(x) - |x|^2/2k \text{ is harmonic in } E_k - \bigcup_m \{2\pi m\}.$$

Furthermore, for $j = 1, \dots, k$,

$$\lim_{|x| \rightarrow 0} H_j(x) + (2\pi)^k x_j / \omega_{k-1} |x|^k \text{ exists and is finite}$$

and

$H_j(x)$ is harmonic in $E_k - \bigcup_m \{2\pi m\}$.

Let us temporarily assume the validity of the lemma, complete the proof of the theorem, and then prove the lemma.

We have that $F(x) + a_0|x|^2/2k$ is harmonic in $E_k - \bigcup_m \{2\pi m\}$ and that $F(x)$ is a periodic function in L^2 on E_k . It is well known [16, p. 269] that under these conditions there is a function $U(x)$ which is harmonic in $B(0, 1)$ such that (with $\Phi(x)$ as in Lemma 8)

$$(50) \quad F(x) + a_0|x|^2/2k = U(x) + c_0'\Phi(x) + \sum_{n=k-1}^{\infty} Y_{n+2-k}(x|x|^{-1})|x|^{-n}$$

for x in $B(0, 1) - 0$ where Y_n is a surface spherical harmonic of order n and the series on the right side of (50) is uniformly convergent outside of any sphere containing the origin. (Note that in two dimensions, using polar coordinates (r, θ) , Y_n is a linear combination of $\sin n\theta$ and $\cos n\theta$.) The fact that $F(x)$ and $\Phi(x)$ are in L^1 on $B(0, 1)$ immediately implies that $Y_n(x)$ is identically zero for $n \geq k$. We conclude, consequently, from (50) that

$$F(x) + a_0|x|^2/2k = U(x) + c_0'\Phi(x) + \sum_{j=1}^k c_j'x_j|x|^{-k} \quad \text{for } x \text{ in } B(0, 1) - 0,$$

and, therefore, from Lemma 8 that there are constants c_0, c_1, \dots, c_k such that

$$(51) \quad \lim_{|x| \rightarrow 0} F(x) + (a_0 - c_0)|x|^2/2k + \sum_{j=0}^k c_j H_j(x) = \beta \text{ where } \beta \text{ is finite-valued.}$$

Setting

$$(52) \quad V(x) = F(x) + (a_0 - c_0)|x|^2/2k + \sum_{j=0}^k c_j H_j(x) \quad \text{for } x \text{ in } E_k - \bigcup_m \{2\pi m\}.$$

$$V(2\pi m) = \beta + (a_0 - c_0)|2\pi m|^2/2k,$$

we obtain from the periodicity of $F(x)$ and $H_j(x)$ and (52) that $V(x)$ is continuous in E_k and consequently from Lemma 8 and the properties already established for F , that $V(x)$ is harmonic in E_k . But $V(x) - (a_0 - c_0)|x|^2/2k$ is periodic and therefore bounded in E_k . We infer, then, from well-known properties of harmonic functions [16, pp. 252-253] that $V(x)$ must be a polynomial of degree less than or

equal to two. However, the only way that the polynomial $V(x) - (a_0 - c_0)|x|^{2/2k}$ can be periodic is if it is a constant. We conclude from (52) that there is a constant K such that

$$(53) \quad F(x) + \sum_{j=0}^k c_j H_j(x) = K \quad \text{for } x \text{ in } E_k - \bigcup_m \{2\pi m\}.$$

From the definition of $F(x)$ and Lemma 8, we obtain from (53) that

$$(54) \quad a_m + c_0 = - \sum_{j=1}^k i c_j m_j \quad \text{for } m \neq 0.$$

By (i) of the theorem, $a_m + c_0 = o(|m|)$ as $m \rightarrow \infty$. We conclude from (50) that $c_j = 0$ for $j = 1, \dots, k$ and therefore from (50) that $a_m = -c_0$. But by (i) of the theorem $\sum_{R \leq |m| \leq R+1} |a_m| = o(R)$ as $R \rightarrow \infty$. On the other hand the number of lattice points in the annulus $R \leq |m| \leq R+1$ is $O(R^{k-1})$ as $R \rightarrow \infty$. c_0 must therefore be zero, and consequently $a_m = 0$ for $m \neq 0$. But then by (ii) of the theorem, $a_0 = 0$, and the theorem is established.

We now prove Lemma 8 (see [24, p. 500]).

We first show that there exists a continuous periodic function $\psi(x)$ such that

$$(55) \quad \begin{aligned} \hat{\Phi}(m) - \hat{\psi}(m) &= |m|^{-2} \quad \text{for } m \neq 0, \\ \hat{\Phi}(0) - \hat{\psi}(0) &= 0. \end{aligned}$$

$$(56) \quad \sum_m |\hat{\psi}(m)| < \infty.$$

Observing from Green's second identity that for $k \geq 3$ and $m \neq 0$

$$\begin{aligned} \int_{T_{k-B}(0, \epsilon)} |x|^{-(k-2)} e^{-i(m, x)} dx \\ = \int_{\partial[T_{k-B}(0, \epsilon)]} e^{-i(m, x)} |m|^{-2} \partial |x|^{-(k-2)} / \partial n dS + o(1) \end{aligned}$$

with a similar remark holding for $\log |x|^{-1}$, we obtain in a manner similar to the establishment of (5) that for $m \neq 0$

$$(57) \quad \begin{aligned} \hat{\Phi}(m) &= |m|^{-2} \\ &+ d_k |m|^{-2} \sum_{j=1}^k \int_{T_{k-1}} \frac{\exp[i(m, x_1 + \dots + m_j x_j^* + \dots + m_k x_k)] \cdot \cos m_j \pi dx_1 \dots dx_j^* \dots dx_k}{[x_1^2 + \dots + \pi^2 + \dots + x_k^2]^{k/2}} \end{aligned}$$

where d_k is a constant depending on k but not on m and $m_j x_j^*$ stands for the deletion of $m_j x_j$.

We now define $\hat{\psi}(m)$ to be the second expression on the right in (57) if $m \neq 0$ and $\hat{\psi}(0) = \hat{\Phi}(0)$. $\hat{\psi}(m)$ consequently satisfies (55). To show (56), we observe that for $m_1 \neq 0$

$$\int_{-\pi}^{\pi} e^{im_1 x_1} [x_1^2 + \dots + \pi^2 + \dots + x_k^2]^{-k/2} dx_1 = k \int_{-\pi}^{\pi} \frac{\sin m_1 x_1 x_1}{m_1 [x_1^2 + \dots + \pi^2 + \dots + x_k^2]^{(k+2)/2}} dx_1$$

and obtain consequently that there is a constant d'_k such that

$$(58) \quad |\hat{\psi}(m)| \leq d'_k (|m| + 1)^{-2} \sum_{j=1}^k (|m_j| + 1)^{-1} \dots \cdot (|m_j| + 1)^{-1*} \dots (|m_k| + 1)^{-1}.$$

On the other hand,

$$(59) \quad [(|m_1| + 1) \dots (|m_k| + 1)]^{(k+1)/k} \leq (|m| + 1)^2 (|m_1| + 1) \dots (|m_j| + 1)^* \dots (|m_k| + 1).$$

(58) and (59) imply that

$$\sum_m |\hat{\psi}(m)| \leq k d'_k \sum_m [(|m_1| + 1) \dots (|m_k| + 1)]^{-(k+1)/k} < \infty,$$

and (56) is established.

We now define $\psi(x) = \sum_m \hat{\psi}(m) e^{i(m,x)}$ and obtain that $\psi(x)$ is a continuous periodic function and furthermore that

$$(60) \quad \Phi(x) - \psi(x) \sim \sum_{m \neq 0} e^{i(m,x)} / |m|^2.$$

Setting $H_0(x, t) = \sum_{m \neq 0} e^{i(m,x) - |m|t} / |m|^2$, it now follows immediately from Theorem 2 that $\lim_{t \rightarrow 0} H_0(x, t) = H_0(x)$ exists and is finite for x in $E_k - \cup_m \{2\pi m\}$ and furthermore that for x in $E_k - \cup_m \{2\pi m\}$, $H_0(x) = \Phi(x) - \psi(x)$. But $\psi(x)$ is a continuous function. Consequently, $\lim_{|x| \rightarrow 0} H_0(x) - \Phi(x)$ exists and is finite.

To complete the part of Lemma 8 dealing with $H_0(x)$, we have to show that $\lim_{t \rightarrow 0} H_0(x, t) = H_0(x)$ in the L^1 -norm on T_k and that $H_0(x) - |x|^2/2k$ is harmonic in $E_k - \cup_m \{2\pi m\}$.

In order to accomplish this, we look at the function $P(x, t) = \sum_m e^{i(m,x) - |m|t}$ for $t > 0$. Using the calculations in Theorem 2, we see that

$$(2\pi)^{-k} \int_{E_k} e^{-i(x,y)-|y|t} dy = b_k t [t^2 + |x|^2]^{-(k+1)/2}$$

and consequently that

$$\int_{E_k} b_k t [t^2 + |x|^2]^{-(k+1)/2} e^{i(y,x)} dx = e^{-|y|t}.$$

But then by Remark 1, (30),

$$b_k (2\pi)^k t \sum_m [t^2 + |x - 2\pi m|^2]^{-(k+1)/2} = P(x, t).$$

We conclude that

$$(61) \quad P(x, t) \geq 0,$$

$$(62) \quad \lim_{t \rightarrow 0} P(x, t) = 0 \text{ uniformly for } x \text{ in } T_k - B(0, \epsilon) \quad \text{for } \epsilon > 0.$$

We next establish the following remark:

REMARK 2. Let f be a function in L^1 on T_k and extended by periodicity to all of E_k . For $t > 0$, set $A_t(f, x) = \sum_m \hat{f}(m) e^{i(m,x) - |m|t}$. Then

$$\lim_{t \rightarrow 0} \int_{T_k} |A_t(f, x) - f(x)| dx = 0.$$

Observing that $A_t(f, x) = (2\pi)^{-k} \int_{T_k} f(x-y) P(y, t) dy$ and that $(2\pi)^{-k} \int_{T_k} P(y, t) dy = 1$, we see that

$$A_t(f, x) - f(x) = (2\pi)^{-k} \int_{T_k} [f(x-y) - f(x)] P(y, t) dy$$

and consequently that

$$\begin{aligned} & \int_{T_k} |A_t(f, x) - f(x)| dx \\ (63) \quad & \leq (2\pi)^{-k} \int_{T_k} P(y, t) \left[\int_{T_k} |f(x-y) - f(x)| dx \right] dy \\ & \leq (2\pi)^{-k} \int_{B(0, \delta)} P(y, t) \left[\int_{T_k} |f(x-y) - f(x)| dx \right] dy + o(1) \end{aligned}$$

by (62) for $0 < \delta < 1$.

Using the fact that $\lim_{|y| \rightarrow 0} \int_{T_k} |f(x-y) - f(x)| dx = 0$, Remark 2 follows immediately from (63).

Since $H_0(x) = \Phi(x) - \psi(x)$ is in L^1 on T_k , it follows from Remark 2 that $\int_{T_k} |H_0(x, t) - H_0(x)| dx \rightarrow 0$ as $t \rightarrow 0$.

Next, let x_0 be a point in $E_k - U_m \{2\pi m\}$. Then there exists $\epsilon > 0$ such that $H_0(x)$ is continuous in $B(x_0, \epsilon)$ since it is equal to $\Phi(x) - \psi(x)$ in this k -ball. By Lemma 7 and (62) $\Delta_* [H_0(x) - |x|^2/2k] \leq 0$ and $0 \leq \Delta^* [H_0(x) - |x|^2/2k]$ for x in $B(x_0, \epsilon)$.

But then by [21, p. 14], $H_0(x) - |x|^2/2k$ is harmonic in $B(x_0, \epsilon)$, and all the statements concerning $H_0(x)$ in Lemma 8 are established.

To establish the rest of the statements in Lemma 8, we note that for $\epsilon > 0$ and small that $\psi(x) + |x|^2/2k$ is harmonic in $B(0, \epsilon) - 0$, and furthermore that $\psi(x) + |x|^2/2k$ is continuous in $B(0, \epsilon)$. Consequently, $\psi(x) + |x|^2/2k$ is harmonic in $B(0, \epsilon)$, and therefore $\lim_{|x| \rightarrow 0} \partial\psi(x)/\partial x_j$ exists and is finite for $j = 1, \dots, k$. But then

$$(64) \quad \lim_{|x| \rightarrow 0} \partial H_0(x)/\partial x_j + (2\pi)^k x_j / \omega_{k-1} |x|^k \text{ exists and is finite.}$$

Also, from what has already been established, we know that $\partial H_0(x)/\partial x_j$ is harmonic in $E_k - U_m \{2\pi m\}$. Furthermore from (64), we observe that $\partial H_0(x)/\partial x_j$ is in L^1 on T_k . Consequently, it follows from Theorem 2 that we shall have established all of Lemma 8 once we show that

$$(65) \quad \int_{T_k} \partial H_0(x)/\partial x_j dx = 0,$$

$$(66) \quad \int_{T_k} e^{-i(m,x)} \partial H_0(x)/\partial x_j = (2\pi)^k i m_j |m|^{-2} \quad \text{for } m \neq 0.$$

To establish (65), let $\mathbf{v}(x)$ be the vector field defined in $T_k - 0$ whose j th component is $H_0(x)$ and whose other $(k-1)$ components are zero. Then by the divergence theorem the left side of (65) is equal to

$$\lim_{\epsilon \rightarrow 0} \int_{\partial[T_k - B(0, \epsilon)]} (\mathbf{v}, \mathbf{n}) dS.$$

But $H_0(x)$ is a periodic function. Consequently $\int_{\partial T_k} (\mathbf{v}, \mathbf{n}) dS = 0$. Therefore the left side of (65) is majorized by

$$\limsup_{\epsilon \rightarrow 0} \max_{|x| = \epsilon} |H_0(x)| \epsilon^{k-1} \omega_{k-1}.$$

However we already have shown that $H_0(x) = o(|x|^{1-k})$ as $|x| \rightarrow 0$. Consequently, (65) is established.

To show that (66) holds, we now let $\mathbf{v}(x)$ be the vector field defined in $T_k - 0$ whose j th component is $e^{-i(m,x)} H_0(x)$ and whose other $(k-1)$ components are zero. Then the same reasoning as above shows that

$$\lim_{\epsilon \rightarrow 0} \int_{\partial\{T_k-B(0,\epsilon)\}} (\mathbf{v}, \mathbf{n})dS = 0.$$

On the other hand, the divergence theorem gives

$$\int_{\partial\{T_k-B(0,\epsilon)\}} (\mathbf{v}, \mathbf{n})dS = \int_{T_k-B(0,\epsilon)} \partial[e^{-i(m,x)}H_0(x)]/\partial x_j dx.$$

We conclude that

$$\begin{aligned} \int_{T_k} e^{-i(m,x)}\partial H_0/\partial x_j dx &= \int_{T_k} im_j H_0(x)e^{-i(m,x)} dx \\ &= im_j(2\pi)^k |m|^{-2} \quad \text{for } m \neq 0. \end{aligned}$$

(66) is established, and the proof of the lemma is complete.

5. Conjugate Fourier series. In this section, we shall describe some results in the theory of conjugate multiple Fourier series and the related topic of analyticity in several variables. The conjugate series will be defined by means of the Calderón-Zygmund kernel. (Other names associated with the development of the singular kernel involved are Beurling, Mihlin, and M. Riesz. See [8, p. 137].)

In order to place the theory in its proper perspective let us briefly review some aspects of the theory of 1-dimensional conjugate Fourier series. If $g(x)$ is a function in L^1 on E_1 , then $\tilde{g}(x) = P.V. g_*x^{-1}$ is defined to be the Hilbert transform of g where

$$P.V. g_*x^{-1} = \lim_{h \rightarrow 0} \pi^{-1} \int_h^\infty [g(x-y) - g(x+y)]/y dy.$$

Now this limit exists almost everywhere, and it is well known that if $\tilde{g}(x)$ is in $L^1 \cap L^2$ then $\widehat{\tilde{g}}(x) = -i \operatorname{sgn} x \hat{g}(x)$ (where $\operatorname{sgn} x = 1$ if $x > 0$, -1 if $x < 0$, and 0 if $x = 0$). However, even if $\tilde{g}(x)$ is not in $L^1 \cap L^2$, we still obtain [34, p. 147] that for almost every x

$$\lim_{R \rightarrow \infty} \int_{-R}^R -i \operatorname{sgn} y \hat{g}(y) e^{ixy} (1 - |y|/R)^\alpha = \tilde{g}(x) \quad \text{for } \alpha > 0.$$

To pass from Fourier integrals to Fourier series, we first observe that [37, p. 73]

$$\frac{1}{2} \cot \frac{x}{2} = x^{-1} + \sum'_{m=-\infty}^{\infty} [(x + 2\pi m)^{-1} - (2\pi m)^{-1}] \quad \text{for } x \neq 2\pi m.$$

Next, we observe that if $f(x)$ is in L^1 on T_1 and extended by periodic-

ity to all of E_1 , then $\tilde{f}(x) = \text{P.V. } f_* 2^{-1} \cot x/2$ is defined to be the conjugate function of f where

$$(67) \quad \text{P.V. } f_* 2^{-1} \cot x/2 = \lim_{h \rightarrow 0} (\pi)^{-1} \int_h^\pi [f(x - y) - f(x + y)] 2^{-1} \cot y/2 dy.$$

Now, it is well known [37, p. 131] that this limit exists almost everywhere, and that if $\tilde{f}(x)$ is in L^1 , then $\widehat{\tilde{f}}(m) = -i \operatorname{sgn} m \hat{f}(m)$. However, even if $\tilde{f}(x)$ is not in L^1 , we still obtain that for almost every x , $\sum_{-\infty}^\infty -i \operatorname{sgn} m \hat{f}(m) e^{imx}$ is Cesaro summable of order α , $\alpha > 0$, to $\tilde{f}(x)$ for almost every x .

We note also that if the limit on the right side of (67) exists then

$$\begin{aligned} \lim_{h \rightarrow 0} \int_h^\pi [f(x - y) - f(x + y)] 2^{-1} \cot y/2 dy \\ = \lim_{h \rightarrow 0} \lim_{\lambda \rightarrow \infty} \int_h^\lambda [f(x - y) - f(x + y)] y^{-1} dy. \end{aligned}$$

Now the function $x^{-1} = \operatorname{sgn} x |x|^{-1}$, and $\operatorname{sgn} 1 + \operatorname{sgn} -1 = 0$. We generalize this function to k -space by means of the kernel

$$(68) \quad K(x) = W(x/|x|) |x|^{-k} \quad \text{for } x \neq 0$$

where $W(x/|x|)$ satisfies the additional conditions:

$$(69) \quad \begin{aligned} \text{there is a constant } \mu \text{ such } |W(\xi_1) - W(\xi_2)| \leq \mu |\xi_1 - \xi_2|^\beta, \\ \beta > 0, \text{ for } \xi_1 \text{ and } \xi_2 \text{ on } C(0, 1) \text{ the unit } (k - 1) \text{ sphere in } E_k \end{aligned}$$

and

$$(70) \quad \int_{C(0,1)} W(\xi) dc(\xi) = 0$$

where $dc(\xi)$ is the natural volume element on $C(0, 1)$.

A kernel $K(x)$ of this type is called a Calderón-Zygmund kernel and is clearly a generalization of the Hilbert kernel x^{-1} in 1-dimension. The generalization persists in the sense that if $g(x)$ is a function in L^1 on E_k , then $\tilde{g}(x) = (2\pi)^{-k} \text{P.V. } g_* K$ is defined to be the Calderón-Zygmund transform of g where

$$\text{P.V. } g_* K(x) = \lim_{\epsilon \rightarrow 0} \int_{E_k - B(0, \epsilon)} g(x - y) K(y) dy.$$

It is shown in [8, p. 118] that this limit exists almost everywhere.

It is also shown in [8] that the principal-valued Fourier transform $\hat{K}(y)$ of the Calderón-Zygmund kernel $K(x)$ exists for every y where

$$\hat{K}(y) = \lim_{\epsilon \rightarrow 0} \lim_{\lambda \rightarrow \infty} (2\pi)^{-k} \int_{B(0,\lambda) - B(0,\epsilon)} e^{-i(x,y)} K(y) dy.$$

This limit is specifically evaluated in [27, p. 69] where it is shown that if $W(\xi)$ has the surface spherical harmonic expansion

$$W(\xi) \sim \sum_{n=1}^{\infty} Y_n(\xi)$$

then for $y \neq 0$,

$$(71) \quad \hat{K}(y) = \sum_{n=1}^{\infty} (-i)^n Y_n(y/|y|) \Gamma(n/2) / 2^k \Gamma[(n+k)/2] \pi^{k/2},$$

where the series in (66) converges absolutely uniformly.

(The special case of (71) when $W(\xi) = Y_n(\xi)$ was first evaluated in [5]. See also [9, p. 261].)

Next, we define $K^*(x)$ the periodic analogue of $K(x)$ as follows:

$$(72) \quad K^*(x) = K(x) + \sum_{m \neq 0} [K(x + 2\pi m) - K(2\pi m)] \quad \text{for } x \neq 2\pi m.$$

In [9], it is shown that the series in (72) converges absolutely uniformly for x in T_k (with a similar fact holding for x in any bounded domain after a finite number of terms of the series are deleted) and that for $x \neq 2\pi m$, $K^*(x + 2\pi m) = K^*(x)$.

If $f(x)$ is in L^1 on T_k and extended by periodicity to all of E_k , it then follows that

$$(73) \quad \tilde{f}(x) = \lim_{\epsilon \rightarrow 0} (2\pi)^{-k} \int_{T_k - B(0,\epsilon)} f(x - y) K^*(y) dy$$

exists and is finite for almost every x .

It is easy to see that if $\int_{T_k} f(y) dy = 0$ and if the limit on the right side of (73) exists and is finite then

$$(74) \quad \tilde{f}(x) = \lim_{\epsilon \rightarrow 0} \lim_{\lambda \rightarrow \infty} (2\pi)^{-k} \int_{B(0,\lambda) - B(0,\epsilon)} f(x - y) K(y) dy.$$

Next, defining the principal-valued m th Fourier coefficient $\hat{K}^*(m)$ as $\lim_{\epsilon \rightarrow 0} (2\pi)^{-k} \int_{T_k - B(0,\epsilon)} e^{-i(m,x)} K^*(x) dx$ it is shown in [9, p. 257] that for $m \neq 0$, $\hat{K}^*(m) = \hat{K}(m)$. It is, furthermore, shown in [9, p. 259] that if $\tilde{f}(x)$ is in L^1 on T_k , then $\tilde{f}^\wedge(m) = \tilde{f}(m) \hat{K}(m)$ for $m \neq 0$. Consequently,

bearing the 1-dimensional situation in mind, it is natural to define $\tilde{S}[f]$, the conjugate series of $S[f]$, as

$$(75) \quad \tilde{S}[f] = \sum_{m \neq 0} \hat{K}(m) \hat{f}(m) e^{i(m,x)}.$$

We shall say that $K(x) = W(x/|x|)/|x|^k$ is a kernel in class C^n if $W(\xi)$ is a function in class C^n on the $(k-1)$ sphere $C(0, 1)$. The following companion results to Theorems 1 and 2 of this survey were then obtained by Shapiro in [27, pp. 43-44].

THEOREM 8. *Let f be in L^1 on T_k and extended by periodicity to all of E_k , and let $S[f] = \sum_m \hat{f}(m) e^{i(m,x)}$. Furthermore let $K(x)$ be a Calderón-Zygmund kernel in class C^{k+4} and let $\hat{K}(y)$ be its principal-valued Fourier transform. Set*

$$\tilde{\sigma}_R^\alpha(f, x) = \sum_{1 \leq |m| \leq R} \hat{K}(m) \hat{f}(m) e^{i(m,x)} (1 - |m|^2/R^2)^\alpha$$

with $(k-1)/2 < \alpha \leq (k-1) + 1/2$ if k is even and $(k-1)/2 < \alpha \leq (k-1)/2 + 1$ if k is odd. Suppose that

$$|B(x_0, h)|^{-1} \int_{B(x_0, h)} |f(x) - f(x_0)| dx \rightarrow 0 \quad \text{as } h \rightarrow 0.$$

Then

$$\lim_{R \rightarrow \infty} \left[\tilde{\sigma}_R^\alpha(f, x_0) - \lim_{\lambda \rightarrow \infty} \int_{B(0, \lambda) - B(0, R^{-1})} f(x_0 - y) K(y) dy \right] = 0.$$

For the 1-dimensional analogue of the above theorem, see [37, p. 95]. We obtain immediately from Theorem 8, (73), and (74) that if $\int_{T_k} f(y) dy = 0$, then for almost every x , $\tilde{\sigma}_R^\alpha(f, x) \rightarrow \tilde{f}(x)$ for $\alpha > (k-1)/2$.

The following companion result for Abel summability (Theorem 2) is also obtained:

THEOREM 9. *Let f be in L^1 on T_k and extended by periodicity to all of E_k and let $S[f] = \sum_m \hat{f}(m) e^{i(m,x)}$. Furthermore, let $K(x)$ be a Calderón-Zygmund kernel in class $C^{k/2}$ or $C^{(k+1)/2}$ according as to whether k is even or odd. Set $\tilde{A}_t(f, x) = \sum_{m \neq 0} \hat{f}(m) \hat{K}(m) e^{i(m,x) - |m|t}$ for $t > 0$. Suppose that $|B(x_0, h)|^{-1} \int_{B(x_0, h)} |f(x) - f(x_0)| dx \rightarrow 0$ as $h \rightarrow 0$. Then*

$$\lim_{t \rightarrow 0} \left[\tilde{A}_t(f, x_0) - \lim_{\lambda \rightarrow \infty} \int_{B(0, \lambda) - B(0, t)} f(x_0 - y) K(y) dy \right] = 0.$$

Actually both Theorems 8 and 9 are proved for a wider class of Calderón-Zygmund kernels than indicated. In particular, Theorem 9 is true for kernels K in class $C^{*\alpha}$, $\alpha > (k-1)/2$. For the definition of

the class $C^{*\alpha}$ and for the proofs of the above theorem, we refer the reader to [27, Chapter II].

Particular cases of what we have called the Calderón-Zygmund kernel were as mentioned earlier previously studied by other individuals. In particular, the kernel $K_j(x) = x_j/|x|^{k+1}$ ($j = 1, \dots, k$) which clearly satisfies (68), (69), and (70) is referred to as the Riesz kernel. (See [32, p. 30].)

The Riesz kernel is quite useful in studying generalizations of the notion of analyticity. In particular if $u(x) = [u_1(x), \dots, u_k(x)]$ is a k -dimensional vector field in class C^1 in a domain D of E_k , we call $u(x)$ an harmonic vector field in D if

$$(76) \quad \operatorname{div} u = 0 \quad \text{and} \quad \operatorname{curl} u = 0 \quad \text{in } D$$

or otherwise stated

$$(77) \quad \sum_{j=1}^k \partial u_j / \partial x_j = 0 \quad \text{and} \quad \partial u_{j_1} / \partial x_{j_2} = \partial u_{j_2} / \partial x_{j_1} \quad j_1 \neq j_2.$$

It is clear that if $k=2$ and u is an harmonic vector field in D , then $u_2 + iu_1$ represents an analytic function in D .

With $K_j(x)$ as above, it follows from (71) that for $y \neq 0$

$$(78) \quad \hat{K}_j(y) = -iy_j/|y| 2^k \Gamma[(k+1)/2] \pi^{(k-1)/2}.$$

Consequently, if $S = \sum_m a_m e^{i(m,x)}$, it is natural to call

$$\tilde{S}_j = - \sum_{m \neq 0} im_j |m|^{-1} a_m e^{i(m,x)}, \quad j = 1, \dots, k,$$

the j th conjugate trigonometric series of S .

Letting $S(x, t) = \sum_m a_m e^{i(m,x) - |m|t}$ and

$$\tilde{S}_j(x, t) = - \sum_{m \neq 0} im_j |m|^{-1} a_m e^{i(m,x) - |m|t},$$

we see that the vector field

$$u(x, t) = [\tilde{S}_1(x, t), \dots, \tilde{S}_k(x, t), S(x, t)]$$

is an harmonic vector field in the variables (x_1, \dots, x_k, t) for $t > 0$.

We shall say that S given above is a Fourier-Stieltjes series if there exist a countably additive set function μ defined on the bounded Borel sets of E_k which is of bounded variation on T_k , which is periodic in the sense that $\mu(A + 2\pi m) = \mu(A)$ for every lattice point m and bounded Borel set A , and which furthermore satisfies

$$a_m = (2\pi)^{-k} \int_{T_k} e^{-i(m,x)} d\mu(x) \quad \text{for every } m.$$

Stein and Weiss [32, p. 55] have succeeded in generalizing the classical theorem of F. and M. Riesz in the context of the above terminology. It is a valid generalization in the sense that if one sets $k=1$ and replaces e^{-t} by r where $0 < t$ and $0 \leq r < 1$, one gets back precisely the theorem of F. and M. Riesz. (See [37, p. 285].) The theorem of Stein and Weiss (though stated in terms of Fourier integrals in [32]) is the following:

THEOREM 10. *Let $S = \sum_m a_m e^{i(m,x)}$ be a Fourier-Stieltjes series due to the set function μ and let $\tilde{S}_j = - \sum_{m \neq 0} i m_j |m|^{-1} a_m e^{i(m,x)}$ be a Fourier-Stieltjes series due to the set function μ_j ($j=1, \dots, k$). Then μ and μ_j ($j=1, \dots, k$) are absolutely continuous set functions. Furthermore if f and f_j are the functions in L^1 on T_k corresponding respectively to μ and μ_j , then there is a constant γ depending on k such that for almost every x*

$$f_j(x) = \lim_{\epsilon \rightarrow 0} \lim_{\lambda \rightarrow \infty} \gamma \int_{B(0,\lambda) - B(0,\epsilon)} f(x-y) y_j |y|^{-(k+1)} dy.$$

There are other notions for generalizing the concept of analyticity than that mentioned above. In particular, there is a method associated with the concept of Dirichlet algebras (see for example [15, p. 54]). This concept leads to the following theorem of Bochner's [4, p. 718] (later reproved by Helson and Lowdenslager [14, p. 184] using different techniques) which is also a generalization of the F. and M. Riesz theorem:

THEOREM 11. *Let $S = \sum_m a_m e^{i(m,x)}$ be a Fourier-Stieltjes series due to the complex-valued set function μ . Suppose there exists a γ with $1 > \gamma > 0$ and an $x_0 \neq 0$ such that $a_m = 0$ if $(m, x_0) < \gamma |m| |x_0|$. Then μ is absolutely continuous.*

6. Riemannian theory. In this section, we shall indicate some of the results in what is called the Riemannian theory of trigonometric series, i.e. the local aspects of trigonometric series.

We shall first look at the theory of formal multiplication as developed by Berkovitz [1] and Shapiro [23] in k -dimensions ($k \geq 2$). These results generalize the 1-dimensional results obtained by Rajchman and others [37, pp. 330-344].

Let $S_1 = \sum a_m e^{i(m,x)}$ and $S_2 = \sum \alpha_m e^{i(m,x)}$ be two trigonometric series with the property that for each m the series $\sum_n |a_m \alpha_{n-m}| < \infty$. We then define the formal product $S_3 = S_1 S_2$ to be the trigonometric series

$$(79) \quad S_3 = \sum_m A_m e^{i(m,x)} \quad \text{where} \quad A_m = \sum_n a_n \alpha_{m-n}.$$

The situation that will occur in the sequel is that the coefficients in S_1 are "bad" in the sense that they grow large as $|m| \rightarrow \infty$ or at least do not go to zero very rapidly and the coefficients in S_2 are "good" in the sense that they go to zero very rapidly.

Given a closed domain \bar{D} contained in the interior of T_k , we shall say $S = \sum_m \alpha_m e^{i(m,x)}$ is a series in class $(S_{\bar{D}})$ if

(80) for every $\theta > n$, there exists an A depending on θ such that

$$|\alpha_m| \leq A(|m| + 1)^{-\theta} \quad \text{for all } m;$$

(81)
$$\sum_m m_1^{j_1} \cdots m_k^{j_k} \alpha_m e^{i(m,x)} = 0 \text{ for } x \text{ in } \bar{D} \text{ and } j_1, \dots, j_k \text{ non-negative integers.}$$

We shall say that $K(x)$ is a spherical harmonic Calderón-Zygmund kernel if

(82)
$$K(x) = P_n(x) |x|^{-(n+k)}$$

where $P_n(x)$ is a homogeneous polynomial of degree $n \geq 1$ and $\Delta P_n(x) = 0$. It then follows from (71) that

(83)
$$\hat{K}(m) = \gamma(n, k) P_n(m) |m|^{-n} \quad \text{for } m \neq 0$$

where $\gamma(n, k) = (-i)^n \Gamma(n/2) / 2^k \Gamma[(n+k)/2] \pi^{k/2}$. In this section, we shall deal only with spherical harmonic Calderón-Zygmund kernels.

The following theorem concerning formal multiplication of trigonometric series then prevails (see [1, p. 326 and p. 330], [23, p. 374]).

THEOREM 12. *Let $S_1 = \sum_m a_m e^{i(m,x)}$ and $S_2 = \sum_m \alpha_m e^{i(m,x)}$ where $a_m = O(|m|^\gamma)$, γ an integer $\geq -(k-1)$, and S_2 is a series of class $(S_{\bar{D}})$, \bar{D} a closed domain contained in the interior of T_k . Set $S_3 = S_1 S_2 = \sum_m A_m e^{i(m,x)}$ and $\tilde{S}_3 = \sum_{m \neq 0} A_m \hat{K}(m) e^{i(m,x)}$ where $K(x)$ is a spherical harmonic Calderón-Zygmund kernel. Then*

- (i) S_3 is uniformly $(B-R, \gamma+k-1)$ summable to zero for x in \bar{D} .
- (ii) \tilde{S}_3 is uniformly $(B-R, \gamma+k-1)$ summable for x in \bar{D} .

As a corollary to the above theorem and method of proof, we obtain the following generalization of the classical theorem of Riemann (see [1] and [23]):

THEOREM 13. *Let $S = \sum_m a_m e^{i(m,x)}$ where $a_m = O(|m|^\gamma)$, γ an integer $\geq -(k-1)$. Set $F(x) = \sum_{m \neq 0} (-1)^\beta (a_m / |m|^{2\beta}) e^{i(m,x)}$ where β is the integral part of $(k+\gamma)/2 + 1$. Suppose that $F(x)$ is class $C^{(2\beta+k+2)}$ in a domain D contained in T_k . Then*

- (i) $\sum_{1 \leq |m| \leq R} a_m e^{i(m,x)} (1 - |m|^2/R^2)^{\gamma+k-1} \rightarrow \Delta^\beta F(x)$ as $R \rightarrow \infty$ uniformly in every closed subdomain of D ,

(ii) $\sum_{1 \leq |m| \leq R} \hat{K}(m) a_m e^{i(m,x)} (1 - |m|^2/R^2)^{\gamma+k-1} \rightarrow$ finite limit as $R \rightarrow \infty$ uniformly in every closed subdomain of D where $K(x)$ is a spherical harmonic Calderón-Zygmund kernel.

In the proof of the uniqueness theorem for trigonometric series we introduced the notion of generalized Laplacians through the use of averages on k -balls. Using averages on $(k-1)$ spheres we can also introduce the notion of generalized Laplacians. (See [22, p. 224].) In particular, we now say that $f(x)$ has a generalized r th-Laplacian at the point x_0 , designated by $\Delta_r f(x_0)$, if [using the notation of (70)] $f(x_0 + t\xi)$ is in L^1 on $C(0, 1)$ the unit $(k-1)$ sphere in E_k for all t sufficiently small and positive and if

$$\omega_{k-1}^{-1} \int_{C(0,1)} f(x_0 + t\xi) dC(\xi) = \sum_{j=0}^r \alpha_j t^{2j} + o(t^{2r}).$$

We then set $\Delta_j f(x_0) = 2^{2j} j! \Gamma(j+k/2) \alpha_j / \Gamma(k/2)$ for $j=0, 1, \dots, r$.

Using this concept of a generalized r th-Laplacian, Shapiro [22] obtained the following result which is a generalization of the classical theorem concerning trigonometric series and generalized derivatives in 1-dimension [38, p. 69]:

THEOREM 14. Let $S = \sum_{m \neq 0} a_m e^{i(m,x)}$ where $a_m = o(|m|^\gamma)$, $\gamma \geq -(k-1)$. A necessary and sufficient condition that there exists an $\alpha \geq 0$ such that

$$\lim_{R \rightarrow \infty} \sum_{1 \leq |m| \leq R} a_m e^{i(m,x)} (1 - |m|^2/R^2)^\alpha = \beta \text{ (finite-valued)}$$

is that there exists an integer $r > \gamma + k - 1$ such that if

$$F(x) = \sum_{m \neq 0} (-1)^r (a_m / |m|^{2r}) e^{i(m,x)}$$

then $\Delta_r F(x_0)$ exists and is equal to β .

For a theorem similar to Theorem 14 for Fourier series generalizing a classical theorem of Hardy and Littlewood [38, p. 69] we have the following result of Chandrasekharan's [10]:

THEOREM 15. Let $f(x)$ be in L^1 on T_k and extended by periodicity to all of E_k , and let $S[f] = \sum_m \hat{f}(m) e^{i(m,x)}$. Then a necessary and sufficient condition that there exists an $\alpha \geq 0$ such that

$$\lim_{R \rightarrow \infty} \sum_{|m| \leq R} \hat{f}(m) e^{i(m,x)} (1 - |m|^2/R^2)^\alpha = \beta \text{ (finite-valued)}$$

is that there exists a $p > 0$ such that

$$A(k, p)t^{-k} \int_{B(0,t)} f(x_0 + y)(1 - |y|^2t^{-2})^{p-1} dy \rightarrow \beta \quad \text{as } t \rightarrow 0$$

where $A(k, p)$ is the constant $[\omega_{k-1} \int_0^1 (1-h^2)^{p-1} h^{k-1} dh]^{-1}$.

7. Geometric integration. In this section, we shall look at some of the applications of multiple trigonometric series to geometric integration theory and potential theory. The future will see many such applications (see the comments in [37, p. 811]).

We shall first discuss an application of multiple trigonometric series to the intrinsic divergence of a vector field. (This result is due to Shapiro [26].)

Let $\mathbf{v}(x) = [v_1(x), \dots, v_k(x)]$ be a continuous vector field defined in a neighborhood of the point x_0 . As before let $B(x_0, t)$ represent the k -ball with center x_0 and radius t , let $C(x_0, t)$ represent the $(k-1)$ -sphere which is its boundary, let $dc(x)$ represent the natural $(k-1)$ -dimensional volume element on $C(x_0, t)$, and let $\mathbf{n}(x)$ represent the outward pointing unit normal vector.

Define

$$\operatorname{div}^* \mathbf{v}(x_0) = \limsup_{t \rightarrow 0} |B(x_0, t)|^{-1} \int_{C(x_0, t)} [\mathbf{v}(x), \mathbf{n}(x)] dc(x).$$

Define $\operatorname{div}_* \mathbf{v}(x_0)$ similar using $\liminf_{t \rightarrow 0}$. In case $\operatorname{div}^* \mathbf{v}(x_0) = \operatorname{div}_* \mathbf{v}(x_0)$ is finite, call this common limit $\operatorname{div} \mathbf{v}(x_0)$. In case, $\operatorname{div} \mathbf{v}(x_0)$ exists, we see that it coincides with the usual definition of the divergence of a vector field defined from an intrinsic point of view. It is furthermore clear that if $\mathbf{v}(x)$ is in class C^1 in a neighborhood of the point x_0 , $\operatorname{div} \mathbf{v}(x_0) = \sum_{j=1}^k \partial v_j(x_0) / \partial x_j$.

Let Q be an open k -cube in E_k and let ∂Q designate its boundary considered as a point set. The divergence theorem in its simplest classical form states that if $\mathbf{v}(x)$ is continuous in \bar{Q} , in class C^1 in Q and if $\operatorname{div} \mathbf{v}(x)$ is in L^1 on Q , then

$$(84) \quad \int_{\partial Q} [\mathbf{v}(x), \mathbf{n}(x)] dc(x) = \int_Q \operatorname{div} \mathbf{v}(x) dx.$$

Let Z be a closed set of measure zero which lies in Q . We define a class C_Z of vector fields as follows:

$\mathbf{v}(x)$ is in C_Z if

- (i) $\mathbf{v}(x)$ is continuous in $\bar{Q} - Z$,
- (ii) $\mathbf{v}(x)$ is in L^2 on Q ,
- (iii) $\operatorname{div}^* \mathbf{v}(x)$ and $\operatorname{div}_* \mathbf{v}(x)$ are finite in $Q - Z$,
- (iv) $\operatorname{div} \mathbf{v}(x)$ exists almost everywhere in Q and is in L^1 on Q .

Because of (i) and (iv), both the left and right sides of (84) are defined. We shall call Z a negligible set for the divergence theorem, if the divergence theorem holds for every v in C_Z , i.e. if $v(x)$ is in C_Z , then (84) is true.

In [26], the following result is obtained:

THEOREM 16. *A necessary and sufficient condition that Z be a negligible set for the divergence theorem is that Z be of capacity zero.*

By capacity zero is meant logarithmic capacity zero in the plane and capacity zero with respect to $|x|^{-(k-2)}$ in E_k for $k \geq 3$.

Condition (ii) above is not artificial because the above theorem is false in E_2 if we widen C_Z by replacing (ii) with " $v(x)$ is in L^p on Q with $1 \leq p < 2$." To see this fact, set $v(x) = \text{grad log } |x|^{-1}$ in $E_2 - 0$ and take $Q = \{x: |x_j| < 1, j=1, 2\}$. Then both sides of (84) are defined but are not equal, and $v(x)$ is in L^p on Q for $1 \leq p < 2$ but is not in L^2 on Q . Furthermore (i), (iii), and (iv) holds with $Z = \{0\}$.

The theorem in multiple trigonometric series which is the key to establishing Theorem 16 is the following [26, p. 611].

THEOREM 17. *Given $S = \sum_m a_m e^{i(m,x)}$ where $\bar{a}_m = a_{-m}$. Let Z be a closed set of capacity zero contained in the interior of T_k . Suppose that the following holds:*

(i) $\sum_{m \neq 0} |a_m|^2 |m|^{-2}$ is finite.

(ii) *There is a function $F(x)$ continuous in the torus sense in $T_k - Z$ and in L^1 on T_k whose Fourier series is given by*

$$S[F] = \sum_{m \neq 0} a_m |m|^{-2} e^{i(m,x)}.$$

(iii) $f^*(x)$ and $f_*(x)$ are finite in $T_k - Z$ and in L^1 on T_k where $f^*(x) = \limsup_{t \rightarrow 0} \sum_m a_m e^{i(m,x) - |m|t}$ and $f_*(x)$ is the corresponding $\liminf_{t \rightarrow 0}$.

Then $f^(x) = f_*(x)$ almost everywhere and S is the Fourier series of $f_*(x)$.*

Theorem 17 is a k -dimensional generalization and extension of a 1-dimensional theorem of Beurling [2]. (For further 1-dimensional results and comments along these lines, see [38, p. 194].)

We next look at an application of multiple trigonometric series to the curl of a vector field defined from an intrinsic point of view and obtain a result which can be viewed as a three-dimensional vector analogue of the classical theorem of Rademacher for functions in Lip 1. (See [20] and [36, p. 371].)

Let $v(x)$ be a three-dimensional vector field defined in a neighborhood of a point x_0 in E_3 . Let $C_n(x_0, r)$ designate the circle with center

x_0 and radius r lying in the plane through x_0 normal to the unit vector \mathbf{n} and oriented by the usual right hand rule. Define the upper circulation per unit area of \mathbf{v} at x_0 in the direction \mathbf{n} , designated by $D_{\mathbf{n}}^*\mathbf{v}(x_0)$, as follows:

$$D_{\mathbf{n}}^*\mathbf{v}(x_0) = \limsup_{r \rightarrow 0} (\pi r^2)^{-1} \int_{C_{\mathbf{n}}(x_0, r)} (\mathbf{v}, \mathbf{t}) ds$$

where \mathbf{t} designates the unit tangent vector and ds the differential of arc length. Similarly, define the lower circulation per unit area, $D_{*\mathbf{n}}\mathbf{v}(x_0)$, using $\liminf_{r \rightarrow 0}$. If $D_{*\mathbf{n}}\mathbf{v}(x_0) = D_{\mathbf{n}}^*\mathbf{v}(x_0)$ and both expressions are finite, designate this common value by $D_{\mathbf{n}}\mathbf{v}(x_0)$ and call it the circulation per unit area of \mathbf{v} at x_0 in the direction \mathbf{n} .

The curl of \mathbf{v} is said to exist at x_0 if $D_{\mathbf{n}}\mathbf{v}(x_0)$ exists for every unit vector \mathbf{n} and if, furthermore, there exists a vector \mathbf{w} such that $(\mathbf{w}, \mathbf{n}) = D_{\mathbf{n}}\mathbf{v}(x_0)$ for every unit vector \mathbf{n} . \mathbf{w} is then called curl $\mathbf{v}(x_0)$.

The curl of \mathbf{v} will be said to exist uniformly at x_0 if curl $\mathbf{v}(x_0)$ exists and if, furthermore,

$$\lim_{r \rightarrow 0} (\pi r^2)^{-1} \int_{C_{\mathbf{n}}(x_0, r)} (\mathbf{v}, \mathbf{t}) ds = \mathbf{n} \cdot \text{curl } \mathbf{v}(x_0)$$

uniformly in \mathbf{n} .

It is clear that if $\mathbf{v}(x)$ is in class C^1 in a neighborhood of the point x_0 , then the curl of \mathbf{v} exists uniformly at the point x_0 . The above definitions are classical and can be found in most of the standard books on advanced calculus or vector analysis.

The following theorem was then established by Shapiro [29]:

THEOREM 18. *Let $\mathbf{v}(x)$ be a continuous vector field defined in an open set R contained in E_3 . Suppose there exists three mutually orthogonal unit vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and a constant K such that $|D_{\mathbf{e}_j}^*\mathbf{v}(x)| \leq K$ and $|D_{*\mathbf{e}_j}\mathbf{v}(x)| \leq K$ for x in R and $j = 1, 2, 3$. Then curl \mathbf{v} exists uniformly almost everywhere in R .*

One of the theorems in multiple trigonometric series on which Theorem 18 depends is the following [26, p. 606]:

THEOREM 19. *Let $\mathbf{v}(x) = [v_1(x), \dots, v_k(x)]$ be a continuous periodic real vector field defined in E_k with $S[v_j] = \sum_m a_m^j e^{i(m, x)}$. Let $f^*(x) = \limsup_{t \rightarrow 0} \sum_m (\sum_{j=1}^k i m_j a_m^j) e^{i(m, x) - |m|t}$ and let $f_*(x)$ be the corresponding $\liminf_{t \rightarrow 0}$. Then for every x ,*

$$\text{div}_* \mathbf{v}(x) \leq f_*(x) \leq f^*(x) \leq \text{div}^* \mathbf{v}(x).$$

For other applications of the theory of multiple trigonometric

series to geometric integration theory, we refer the reader to [27] and [28].

We close this survey with the comment that the theory of multiple trigonometric series is yet in its infancy.

8. Bibliographies.

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