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ON APPROXIMATE SOLUTIONS TO THE CONVOLUTION EQUATION ON THE HALF-LINE

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1. We will call a complex-valued function on the half-line $t>0$ locally integrable if it is integrable on each interval $[0, T]$, $T>0$. Let $\mathcal{L}$ be the ring of locally integrable functions (functions which are equal up to a set of measure zero will be identified with each other) with the usual pointwise addition, and with convolution for the product operation. Thus $kx=r$ if and only if $\int_0^t k(t-u)x(u)du=r(t)$ for almost every $t>0$. Give $\mathcal{L}$ the topology defined by the seminorms $\|x\|_T=\int_0^T|x|(u)du$, $T>0$. Thus a sequence x_n, $n=1, 2, \cdots$ in $\mathcal{L}$ converges to 0 in $\mathcal{L}$ if and only if $x_n\to 0$ in $L[0, T]$ for each $T>0$ as $n\to\infty$. The equation $kx=r$ is an important integral equation; however, solutions and the existence of solutions are in general difficult to obtain. M. I. Fenyö and C. Foias [1][1] have shown that if k and r are in $\mathcal{L}$ and if k vanishes on no neighborhood of the origin (i.e. $\|k\|_T>0$ for each $T>0$) there is always an approximate solution to the equation

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[1] The author thanks the referee for calling this article to his attention.

$kx=r$ in the sense that if $T>0$ and $\epsilon>0$ are given there is an x in $L[0, T]$ such that $\|r-kx\|_T<\epsilon$. We shall give a new proof of this result which enables one to see how such approximate solutions can be constructed, when k is a real function, in terms of the characteristic functions of a completely continuous self-adjoint operator on a Hilbert space.

2. Each element k in $\mathfrak{L}$ defines a continuous linear transformation of $\mathfrak{L}$ into $\mathfrak{L}$, and of $L^p[0, T]$ into $L^p[0, T]$ (for each $T>0$, $1\leqq p\leqq\infty$) by the formula $K(x)=kx$. Let $\overline{K}$ be the transformation defined by the complex conjugate $\bar{k}$ of k, and denote by S the transformation of $L^p[0, T]$ into $L^p[0, T]$ which is defined by $S(x)(t)=x(T-t)$ for all t in $[0, T]$. If K is considered as an operator on the space $L[0, T]$ it is easy to verify that the adjoint transformation of K is $K^*=S\overline{K}S$. By a well-known theorem of Titchmarsh if k does not vanish on a neighborhood of the origin then $K(x)=0$ in $L[0, T]$ if and only if x is the zero element of $L[0, T]$; thus the null space of $K^*=S\overline{K}S$ consists of the zero element alone and the range of K is dense in $L[0, T]$.

We will call a sequence x_n, $n=1, 2, \cdots$, in $\mathfrak{L}$ an approximate solution to the equation $kx=r$ if $kx_n\rightarrow r$ in $\mathfrak{L}$ as $n\rightarrow\infty$. We have proved the following theorem.

THEOREM (FOIAŞ). *If k in $\mathfrak{L}$ vanishes on no neighborhood of the origin and r is in $\mathfrak{L}$ there is an approximate solution in $\mathfrak{L}$ to the equation $kx=r$.*

3. Henceforth we shall consider k to be real and to vanish on no neighborhood of the origin.[2] In order to construct approximate solutions we consider K and S as operators on the Hilbert space $L^2[0, T]$. The operator KS is self-adjoint since $(KS)^*=S^*K^*=S(SKS)=KS$. Moreover, if B is any bounded set in $L^2[0, T]$ the set $\{kx \mid x \text{ in } B\}$ is bounded and is equicontinuous in norm; thus, it has compact closure. It follows that K is completely continuous, and consequently KS is a completely continuous self-adjoint operator on the Hilbert space $L^2[0, T]$. Let λ_n, $n=1, 2, \cdots$, be the characteristic values of KS and let ϕ_n, $n=1, 2, \cdots$, be the corresponding orthonormal characteristic functions. Since the null space of KS consists of the zero element alone, the characteristic functions ϕ_n form a complete orthonormal system for $L^2[0, T]$. The equation $kx=f$ with k in $L[0, T]$ and f in $L^2[0, T]$ has a solution in $L^2[0, T]$ if and only if

[2] If k is not real, i.e. KS is not self-adjoint, essentially these same methods can be used. See F. Smithies, *Integral equations*, Cambridge Tracts in Mathematics and Mathematical Physics, No. 49, Chapter VIII.

$\sum_0^\infty |(f, \phi_n)/\lambda_n|^2 < \infty$, and if this quantity is finite that solution is given by

$$x = \sum_0^\infty \frac{(f, \phi_n)}{\lambda_n} S(\phi_n).$$

Even if there is no solution in $L^2[0, T]$ the functions

$$x_{f,N} = \sum_0^N \frac{(f, \phi_n)}{\lambda_n} S(\phi_n) \qquad N = 1, 2, \cdots$$

are such that $K(x_N) = \sum_0^N (f, \phi_n)\phi_n \to f$ in $L^2[0, T]$ and a fortiori in $L[0, T]$ as $N \to \infty$. We can now construct an approximate solution to the equation $kx = r$. For each positive integer i pick f_i in $L^2[0, i]$ such that $\|r - f_i\|_i < 1/i$, and for each i take N_i such that $\|f_i - x_{f_i,N_i}\|_i < 1/i$. The functions x_i which are such that $x_i(t) = x_{f_i,N_i}(t)$ on $[0, i]$ and $x_i(t) = 0$ for $t > i$ constitute an approximate solution to the equation $kx = r$.

4. For $0 < \alpha < \infty$ let $\mathfrak{L}_\alpha = \{x \mid x \in \mathfrak{L} \text{ and } \|x\|_\alpha = 0\}$.

COROLLARY 1. *A is a closed proper ideal in $\mathfrak{L}$ if and only if $A = \mathfrak{L}_\alpha$ for some α.*

PROOF. Clearly each $\mathfrak{L}_\alpha$ is a closed proper ideal. If A is a closed proper ideal in $\mathfrak{L}$ let $\bar{\beta} = \inf\{\beta \mid \beta > 0, \exists x \in A, \|x\|_\beta > 0\}$. $\bar{\beta}$ is a non-negative number, and by the above theorem A, being closed, contains $\mathfrak{L}_\beta$ for each $\beta > \bar{\beta}$. Since A is closed and not equal to $\mathfrak{L}$, $\bar{\beta}$ is not zero and $A = \mathfrak{L}_{\bar{\beta}}$.

In particular there are no proper maximal ideals in $\mathfrak{L}$ or in any of the Banach algebras $L[0, T]$, $T > 0$. This yields a theorem of Ryll-Nardzewski [2]:

COROLLARY 2. *For $f \in \mathfrak{L}$, $f^n \to 0$* in *$\mathfrak{L}$ as $n \to \infty$.*

PROOF. Since $L[0, T]$ has no maximal ideals the spectral radius of $f \in L[0, T]$ equals zero, and thus $\|f^n\|_T \to 0$ as $n \to \infty$ for each $T > 0$.

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