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SIMPLY INVARIANT SUBSPACES[1]

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Let L^1, L^2 denote respectively the spaces of summable and square summable functions on the circle group and H^1, H^2 their subspaces consisting of those functions whose Fourier coefficients vanish for negative indices. A closed subspace M of L^1 or L^2 is "invariant" if

$$\chi M \subset M$$

and "simply invariant" if the above inclusion is strict, where χ is the character

$$\chi(x) = e^{ix}.$$

The structure of simply invariant subspaces is known, namely, they are precisely the subspaces of the form qH^1 or qH^2 (respectively) where q is a measurable function of modulus 1 a.e. Beurling [1] first proved this for subspaces $M \subset H^2$; for $M \subset H^1$, this is due to de Leeuw-Rudin [5]; for $M \subset L^2$, due to Helson-Lowdenslager [3] and for $M \subset L^1$, due to Forelli [2]. In [3] Helson-Lowdenslager also gave a simple proof of the H^2 case, free of function theoretic considerations. Using their arguments Hoffman [4] extended this result to *simply invariant* subspaces of $H^2(dm)$ defined over *logmodular algebras*. In this paper we prove this result for simply invariant subspaces of $L^2(dm)$ and $L^1(dm)$ over logmodular algebras; the results of the previous authors follow as a corollary. The proofs of the previous authors

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[1] This work was done while I held a visiting appointment at the University of California, Berkeley.

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do not extend to this general case as they depend on facts which either have no analogues or are not true for the logmodular algebras; when specialised to their contexts, our proof turns out to be even simpler. Our proof for the case of $L^2(dm)$ was inspired by that of Helson-Lowdenslager for the H^2 case and is in the same spirit as theirs.

Let X be a compact Hausdorff space and $\boldsymbol{A}$ a subalgebra of the algebra $C(X)$ of complex continuous functions on X with the uniform norm.

$\boldsymbol{A}$ is *logmodular* if

i. $\boldsymbol{A}$ is uniformly closed,

ii. $\boldsymbol{A}$ contains the constant functions,

iii. $\boldsymbol{A}$ separates the points of, X and

iv. the set of functions $\log |f|$ where f, $1/f \in \boldsymbol{A}$, is uniformly dense in the algebra of real continuous functions on X.

Let m be a probability Baire measure on X which is "multiplicative" on $\boldsymbol{A}$, meaning

$$\int fg\, dm = \int f\, dm \int g\, dm$$

for all f, $g \in A$ (such measures always exist), and let $\boldsymbol{H}^1(dm)$, $\boldsymbol{H}^2(dm)$ denote the closures of $\boldsymbol{A}$ in $\boldsymbol{L}^1(dm)$, $\boldsymbol{L}^2(dm)$ respectively. The *invariant* subspaces $\boldsymbol{M}$ are now closed subspaces of $\boldsymbol{L}^1(dm)$, $\boldsymbol{L}^2(dm)$, which are invariant under multiplication by functions in $\boldsymbol{A}$ or equivalently by functions in $\boldsymbol{A}_0$, where

$$\boldsymbol{A}_0 = \left\{ f \,\middle|\, f \in A, \int f\, dm = 0 \right\}$$

and the *simply invariant* $\boldsymbol{M}$'s are those for which the inclusion $\boldsymbol{A}_0\boldsymbol{M} \subset \boldsymbol{M}$ is strict.[2]

In the case considered earlier, X was the unit circle, $\boldsymbol{A}_0$ was the uniform closure of the algebra generated by χ in $C(X)$ and m the normalised Lebesgue measure. We have

THEOREM.

1. *The simply invariant subspaces of $\boldsymbol{L}^2(dm)$ are precisely the subspaces of the form $q\boldsymbol{H}^2(dm)$ where $q \in \boldsymbol{L}^2(dm)$ and $|q| = 1$ a.e. (dm).*

2. *The simply invariant subspaces of $\boldsymbol{L}^1(dm)$ are precisely the subspaces of the form $q\boldsymbol{H}^1(dm)$ where $q \in \boldsymbol{L}^1(dm)$ and $|q| = 1$ a.e. (dm).*[3]

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[2] A_0M should be replaced by its closure in $L^2(dm)$ respectively $L^1(dm)$, which necessitates changes in the proof.

[3] The details of the proof of the L^1 theorem and its function theoretic consequences will be published separately.

PROOF. It is obvious that subspaces of the form $qH^2(dm)$, $qH^1(dm)$ are invariant; they are simply invariant because for instance, $q\in qH^2(dm)$, $qH^1(dm)$ while $q\notin qA_0H^2(dm)$, $qA_0H^1(dm)$. To prove the converse:

1. We need the following facts about logmodular algebras [4, pp. 284, 293]:

(a) $A+\bar{A}$ is dense in $L^2(dm)$ where the bar denotes complex conjugation,

(b) if μ is any positive Baire measure on X such that $\int f d\mu=0$ for all $f\in A_0$ then $d\mu=c\,dm$ for some constant c.

Now let $M\subset L^2(dm)$ be simply invariant and let $q\in M\ominus A_0M$, $q\neq 0$. Then $q\perp A_0q$, so $\int f|q|^2dm=0$ for all $f\in A_0$ and by (b), $|q|^2=c$ a.e. By modifying q we may assume that $|q|=1$ a.e.

Clearly $qH^2(dm)\subset M$, because of invariance of M. Let $g\in M\ominus qH^2(dm)$. Then $g\perp qA$, so $g\bar{q}\perp A$. Also $A_0g\subset A_0M$, so $q\perp A_0g$ so that $g\bar{q}\perp \bar{A}_0$. Thus $g\bar{q}\perp A+\bar{A}$, hence $g\bar{q}=0$ a.e. by (a) and since $|q|=1$ a.e., $g=0$. Thus $M=qH^2(dm)$.

2. We use (1) to prove (2). Let $N\subset L^1(dm)$ be simply invariant and let $M=N\cap L^2(dm)$. M is clearly an invariant subspace of $L^2(dm)$. We shall show that it is actually simply invariant. Let $f\in N$. We can find $f_1, f_2\in L^2(dm)$ such that $f=f_1f_2$; we may also assume that one of them, say, f_2 is nonzero a.e. Then $f_2H^2(dm)$ is a simply invariant subspace of $L^2(dm)$ and is by (1) of the form $q_2H^2(dm)$, $|q_2|=1$ a.e. Now

$$f_1q_2\in f_1q_2H^2(dm) = f_1f_2H^2(dm) = fH^2(dm)\subset N.$$

Also $f_1q_2\in L^2(dm)$. Hence $f_1q_2\in M$. Suppose $M=A_0M$. Then $f_1q_2\in A_0M$. Let

$$f_1q_2 = f_0g, \qquad f_0\in A_0, \qquad g\in M\subset N$$

and

$$f_2 = q_2h, \qquad h\in H^2(dm).$$

Then

$$f = f_1f_2 = f_1q_2h = f_0gh \in A_0NH^2(dm)\subset A_0N$$

and it follows that $N=A_0N$. Hence if N is simply invariant, so is M.

Let then $M=qH^2(dm)$ by (1). We shall show that $N=qH^1(dm)$. Clearly $qH^1(dm)\subset N$. Let $f\in N$ and f_1, f_2, q_2, h be as above. Then $f_1q_2\in M=qH^2(dm)$. Let $f_1q_2=gh'$, $h'\in H^2(dm)$. Then

$$f = f_1f_2 = f_1q_2h = qh'h\in qH^1(dm)$$

as h', $h\in H^2(dm)$. It follows that $N=qH^1(dm)$.

We may remark that if $M \subset \boldsymbol{H}^2(dm)$ is invariant and we assume with Hoffman [4, p. 293] that $\int g dm \neq 0$ for at least one $g \in \boldsymbol{M}$ then $\boldsymbol{M}$ is certainly simply invariant and Hoffman's result follows. But this latter condition is not necessary for simple invariance as the example of $z^k \boldsymbol{H}^2$, $k \geqq 1$ shows.

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