THE FUNDAMENTAL GROUP AND THE FIRST COHOMOLOGY GROUP OF A MINIMAL SET

BY HSIN CHU AND MICHAEL A. GERAGHTY[1]

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1. **Introduction.** A conjecture of long standing, posed by Professor W. H. Gottschalk, is whether an n-sphere cannot be a minimal set under a continuous flow, for an odd n greater than one. More generally, must a compact manifold which is minimal under a continuous flow have a nontrivial fundamental group. Or more generally yet, must a compact Hausdorff space which is minimal under a continuous flow have a nontrivial first integral cohomology group in the sense of Alexander-Wallace-Spanier.

Now let X be a locally pathwise-connected compact Hausdorff space such that every map of X into S^1 is homotopic to a constant, i.e. $\pi(X, S^1)=0$. Then it is shown in this paper that if X is minimal under a continuous flow, X must be totally minimal. The above questions then reduce to the existence of totally minimal flows.

It is further shown that for such spaces X a totally minimal flow cannot be locally almost periodic. So on such spaces one cannot have a locally almost periodic minimal flow.

In particular, for a sphere or real projective space of odd dimension greater than one or for a lens space, if it is a minimal set under a continuous flow, then it is totally minimal and so it is not locally almost periodic. For terminology we refer to Gottschalk-Hedlund [**4**]. Incidentally, the above results constitute a partial answer to Problem 1 of [**5**].

2. **The main theorem.** In the case of compact Hausdorff spaces X, it is known that $\pi^1(X)$ the Bruschlinsky group, which as a set is just $\pi(X, S^1)$, is isomorphic to the first integral A-W-S cohomology group $H^1(X)$. (See [**6**].) So either of these groups being zero implies that $\pi(X, S^1)$ is zero and every map of X to S^1 induces the zero homomorphism on $\pi_1(X)$. We may also derive this conclusion from the assumption that $\pi_1(X)$ has no factor group isomorphic with the integers.

THEOREM. *Let X be a compact, Hausdorff, locally pathwise-connected space such that for any map f from X to S^1, the induced homomorphism f_* on $\pi_1(X)$ is trivial. Then if X is a minimal set under a continuous flow, X is totally minimal.*

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PROOF. Let (X, R, π) be the given transformation group. Now since (X, R, π) is not totally minimal, there exists a closed syndetic subgroup G such that for all points b in X the orbit closure $\mathrm{Cl}(\pi(b, G))$ is not all of X. Since we are concerned only with the additive structure of R, we may assume without loss of generality that G is the subgroup Z of the integers of R.

Let Q_1 be the relation on X defined by the orbit closures of Z, namely xQ_1y if x is in $\mathrm{Cl}(\pi(y, Z))$. Q_1 is an open and a closed relation (see [4, Chapter II]) and X is normal, and under these conditions one can easily see that the quotient space $X^* = X/Q_1$ is Hausdorff. Let $p_1: X \to X^*$ be the quotient map.

Denote by Q_2 the usual "modulo one" relation on the reals R and let $p_2: R \to S^1$ be the quotient map to the unit circle, denoted by the reals modulo one. We have that $\pi: X \times R \to X$ maps the relation $Q_1 \times Q_2$ into the relation Q_1 and so induces a map on $(X \times R)/(Q_1 \times Q_2)$. Now Q_1 and Q_2 are both open, and so we may identify $(X \times R)/(Q_1 \times Q_2)$ and $(X/Q_1) \times (R/Q_2)$. Thus π induces a continuous function π^* and the diagram

$$\begin{array}{ccc} X \times R & \xrightarrow{\pi} & X \\ \Big\downarrow{\scriptstyle p_1 \times p_2} & & \Big\downarrow{\scriptstyle p_1} \\ X^* \times S^1 & \xrightarrow{\pi^*} & X^* \end{array}$$

is commutative. It follows easily that (X^*, S^1, π^*) is a transformation group.

For the remainder of the paper, let b be any chosen base point in X. Define $i: R \to X \times R$ by $i(r) = (b, r)$. Consider the mapping

$$R \xrightarrow{i} X \times R \xrightarrow{\pi} X \xrightarrow{p_1} X^*.$$

By assumption, (X, R, π) is minimal, and so $\pi \circ i(R)$ is dense in X and thus $p_1 \circ \pi \circ i(R)$ is dense in X^*.

Next consider the map

$$R \xrightarrow{i} X \times R \xrightarrow{p_1 \times p_2} X^* \times S^1 \xrightarrow{\pi^*} X^*.$$

Then $(p_1 \times p_2) \circ i(R) = \{p_1(b)\} \times S^1$ is homeomorphic to S^1. So $\pi^* \circ (p_1 \times p_2) \circ i(R)$ is the continuous image in X^* of the compact set $\{p_1(b)\} \times S^1$ and since X^* is Hausdorff this image is compact while by the above commutative diagram, it is also dense in X^*. So

$$\pi^* \circ (p_1 \times p_2) \circ i(R) = p_1 \circ r \circ i(R) = X^*.$$

Now since by assumption $\mathrm{Cl}(\pi(b, Z))$ is not all of X, X^* does not reduce to a point. In the action (X^*, S^1, π^*) let K be the isotropy subgroup of $p_1(b)$. Then K is a closed and proper and so finite subgroup of S^1. Let $Z_0 = p_2^{-1}(K)$ in R. Then the above construction of (X^*, S^1, π^*) may be repeated for the subgroup Z_0 of R. It follows that the new isotropy subgroup will be the identity in S^1. So by proper choice of the subgroup Z_0 in R, we may assume that the isotropy subgroup is trivial. Again without loss of generality we may assume that Z_0 is Z.

We may then define $j: X^* \times S^1 \to S^1$ by $j(x^*, s) = s$, and we have the diagram

$$\begin{array}{ccccc} R & \xrightarrow{i} & \{b\} \times R & \xrightarrow{\pi} & X \\ & & \downarrow p_1 \times p_2 & & \downarrow p_1 \\ S^1 & \xleftarrow{j} & \{p_1(b)\} \times S^1 & \xrightarrow[\pi_b^*]{} & X^*, \end{array}$$

where π_b^* is the restriction of π^* to the $p_1(b)$-fibre. Define $g = j \circ (\pi_b^*)^{-1} \circ p_1 \circ \pi \circ i$. Then by commutativity, $g = j \circ (p_1 \times p_2) \circ i = p_2$.

We shall need the following:

DEFINITION. Let $f: A \to S^1$ be a map, where A is a closed interval of R, one endpoint of which is zero and the other a real number r. Let $p_2: R \to S^1$ be the usual quotient map, with the usual orientations. Now f may be lifted to a map $F: A \to R$ with $f(0)$ lifted to $F(0)$ in the interval $[0, 1)$ in a unique orientation preserving fashion. Define W. N. of $f(r)$ = *Winding Number of* $f = F(r) - F(0)$. Note that W. N. of p_2 at r is r, for all real r.

Now the key to the proof is to show that while g and p_2 are identical by commutativity, their winding numbers are distinct.

Define $m = j \circ (\pi_b^*)^{-1} \circ p_1: X \to S^1$. Then by assumption, since $m_*(\pi_1(X)) = 0$, the degree of m is zero. Choose $0 < \delta < \frac{1}{2}$, and let α be an index in X such that for (x, y) in α, $d(m(x), m(y)) < \delta$, where d is the usual distance on the reals modulo one. Let β^2 be in α, and let V be a pathwise-connected neighborhood of b such that V is in $\beta(x)$. Then for y and z in V,

(a) there is a path $\sigma: I \to V$, $I = [0, 1]$, where $\sigma(0) = y$, $\sigma(1) = z$;

(b) the winding number of the function $m \circ \sigma: I \to S^1$ is less than $2\delta < 1$ in absolute value.

Now let $\pi(b, r)$, for some $r > 0$, be any later point of the orbit of b which lies in V. Such exist since X is compact minimal. Then $\pi \circ i$: $R \to X \times R \to X$, restricted to the interval $[0, r]$, is a path μ in X from b to $\pi(b, r)$. Let σ be a path in V from $\pi(b, r)$ to b.

Let $T=[0, r]\cup[0, 1]$, the union space. Define $\Sigma: T\to X$ by $\Sigma=\mu\cup\sigma$. Now S^1 is a quotient space of T, upon identifying end-points, and Σ induces a map $\Sigma^*: S^1\to X$. Then

$$m\circ\Sigma^*: S^1\to S^1.$$

But this map factors through X. So $m\circ\Sigma^*$ induces a trivial map from $\pi_1(S^1)$ to $\pi_1(S^1)$.

Now the degree of $m\circ\Sigma^*$ is the sum of the winding numbers of $m\circ\Sigma^*$ restricted to the images of $[0, r]$ and $[0, 1]$ with the usual orientation. But this degree must be zero. So the winding number of $m\circ\mu$ is the negative of the winding number of $m\circ\sigma$.

Then $|\text{W. N. of } m\circ\mu(r)|<2\delta<1$. But $m\circ\mu=g$ on $[0, r]$. So $|\text{W. N. of } g(r)|<2\delta<1$. Now since X is compact minimal there is some $r_0>1$ in R such that $\pi(b, r_0)$ is in V. Then

$$|\text{W. N. of } g(r_0)| < 1,$$

while W. N. of $p_2(r_0)=r_0>1$.

The main theorem follows from this contradiction. There is an alternate proof using the covering map property (see [6]) on the above diagram.

3. Some corollaries. We need the following:

LEMMA. *Let (X, T, π) be a transformation group, where X is compact, Hausdorff and T is a locally compact, non-totally-disconnected, abelian group. Let T_0 be the connected component of the identity in T and assume that a in X is such that $\pi(a, T_0)\neq a$. Let X be minimal and locally almost periodic under T. Then X is not totally minimal.*

PROOF. By Theorem 10.07 of [4], we have for any b in $\pi(a, T_0)$, $b\neq a$, that a and b are distal. Denoting by P the proximal relation, since X is locally almost periodic we have that P is a closed equivalence relation (see [3]) and the induced transformation group $(X/P, T, \pi^*)$ is almost periodic minimal. Also $aP\neq bP$. So by the theorem of [2], X/P is not totally minimal, and so neither is X.

The completion of this proof was aided by a conversation with J. Auslander.

We now have the following four corollaries.

COROLLARY 1. *No sphere or real projective space of dimension greater than one, nor any lens space, can be a locally almost periodic minimal set under a continuous flow.*

PROOF. Immediate from the main theorem and the lemma.

COROLLARY 2. *Let* (X, R, π) *be a transformation group where* R *is the real numbers and* X *is compact, Hausdorff, locally pathwise-connected, minimal and not totally minimal. Then* $\pi(X, S^1) = \pi^1(X) \approx H^1(X) \neq 0$, *and* $\pi_1(X)$ *must have a factor group isomorphic to the integers.*

PROOF. A restatement of the main theorem.

COROLLARY 3. *Let* (X, R, π) *be a transformation group where* R *is the real numbers and* X *is a compact Hausdorff orientable manifold, minimal and not totally minimal. Then* $H_1(X)$, $H^1(X)$ *and* $\pi^1(X)$ *have factor groups isomorphic to the integers.*

PROOF. In this case we have $H_1^w(X) \approx H_w^1(X)$ while $H_1(X)$ is $\pi_1(X)$ modulo its commutator subgroup. The corollary follows from Corollary 2.

COROLLARY 4. *If* G *is a nontrivial, separable, connected, locally pathwise-connected, compact Hausdorff abelian group, then* $H^1(G)$ *is nontrivial and* $\pi_1(G)$ *has a factor group isomorphic to the integers.*

PROOF. Every such group is almost periodic minimal under some continuous flow. (See [1].)

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UNIVERSITY OF ALABAMA