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THE PRODUCT OF A NORMAL SPACE AND A METRIC SPACE NEED NOT BE NORMAL

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An old—and still unsolved—problem in general topology is whether the cartesian product of a normal space and a closed interval must be normal. In fact, until now it was not known whether, more generally, the product of a normal space X and a metric space Y is always normal. The purpose of this note is to answer the latter question negatively, even if Y is separable metric and X is Lindelöf and hereditarily paracompact.

Perhaps the simplest counter-example is obtained as follows: Take Y to be the irrationals, and let X be the unit interval, retopologized to make the irrationals discrete. In other words, the open subsets of X are of the form $U \cup S$, where U is an ordinary open set in the interval, and S is a subset of the irrationals.[2] It is known, and easily verified, that any space X obtained from a metric space in this fashion is normal (in fact, hereditarily paracompact). Now let Q denote the rational points of X, and U the irrational ones. Then in $X \times Y$ the two disjoint closed sets $A = Q \times Y$ and $B = \{(x, x) \mid x \in U\}$ cannot be separated by open sets. To see this, suppose that V is a neighborhood of B in $X \times Y$. For each n, let

$$U_n = \{x \in U \mid (\{x\} \times S_{1/n}(x)) \subset V\},$$

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[1] Supported by an N.S.F. contract.

[2] The usefulness of this space X for constructing counterexamples was first called to my attention, in a different context, by H. H. Corson.

where $S_{1/n}(x)$ denotes the $1/n$-sphere about x in Y. The U_n cover U, and since U is not an F_σ in X, there exists an index k such that $\overline{U}_k \cap Q \neq \emptyset$. Pick an x in $\overline{U}_k \cap Q$, and then pick $y \in Y$ such that $|x-y| < 1/2k$. Then $(x, y) \in A$, and we need only show that any rectangular neighborhood $R \times S$ of (x, y) in $X \times Y$ intersects V. To do that, pick $x' \in R \cap U_k$ so that $|x'-x| < 1/2k$. Then $(x', y) \in R \times S$; also

$$|x' - y| \leqq |x' - x| + |x - y| < \frac{1}{2k} + \frac{1}{2k} = \frac{1}{k},$$

so $(x', y) \in V$ because $x' \in U_k$. That completes the proof.[3]

The space X in the above example is neither Lindelöf nor separable. If Lindelöf is desired, let Y' be an uncountable subset of the unit interval, all of whose compact subsets are countable; such spaces exist [1, p. 422]. Letting X' be the unit interval, retopologized to make Y' discrete, we see just as before that X' is hereditarily paracompact and that $X' \times Y'$ is not normal; moreover, because of the peculiar property of Y', it is easily checked that X' is Lindelöf.[4] This X' is still not separable; it can, however, be embedded as a closed subset of a separable, Lindelöf, paracompact space X'', and then $X'' \times Y'$ is also not normal.[5]

Note that none of the above spaces X, X', and X'' are—or could be—perfectly normal, since the product of a paracompact, perfectly normal space and a metrizable space is known to be paracompact [2, Proposition 5]. That explains why none of our X's are either *hereditarily* Lindelöf, or separable and *hereditarily* paracompact, since—as is not hard to see—that would make them perfectly normal.

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[3] The above construction remains valid if Y is any separable metric space which is not σ-compact, or, even more generally, any metric space which can be embedded as a non-F_σ subset in another metric space. For instance, Y may be any infinite-dimensional Banach space.

[4] If the continuum hypothesis is assumed, one can even find a Lindelöf, hereditarily paracompact space whose product with the irrationals is not normal.

[5] Observe that both X and X'—but not X''—have a σ-disjoint (and hence point-countable) base.