A TOPOLOGICAL CLASSIFICATION OF CERTAIN 3-MANIFOLDS

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Introduction. In [1] J. Stallings proves that members of the class of closed irreducible 3-manifolds which are fibered over a circle by an aspherical 2-manifold may be distinguished from other closed irreducible 3-manifolds by their fundamental group alone.

He asks whether two members of this class of 3-manifolds are homeomorphic if they have isomorphic fundamental groups. This question is answered in the affirmative here, thus giving a classification of these manifolds according to their fundamental group.

The closed case. Let us denote by $\mathfrak{M}$ the class of all 3-manifolds satisfying the following conditions:

(a) Manifolds of $\mathfrak{M}$ are irreducible (every 2-sphere bounds a 3-cell).

(b) Manifolds of $\mathfrak{M}$ are closed.

(c) Manifolds of $\mathfrak{M}$ have fundamental groups which contain a finitely generated normal subgroup of order >2, with quotient group an infinite cyclic group.

THEOREM 1. *Let M_2 be any closed irreducible 3-manifold. Let M_1 belong to $\mathfrak{M}$, then M_1 is homeomorphic to M_2 if and only if $\pi_1(M_1)$ is isomorphic to $\pi_1(M_2)$.*

PROOF. One direction is trivial. By Stallings theorem [1] M_1 admits a fibering over S^1, with fiber a closed 2-manifold T_1. Let

$$1 \to H_1 \to G_1 \to Z \to 0 \tag{1}$$

denote the sequence of fundamental groups of T_1, M_1, S^1, respectively corresponding to this fibering. Let ρ^* denote the assumed isomorphism from $\pi_1(M_1) = G_1$ to $\pi_1(M_2) = G_2$. Then ρ^* induces

$$1 \to H_2 \to G_2 \to Z \to 0. \tag{2}$$

Now, G_1 and G_2 are both described by giving the automorphisms ϕ_1^*, ϕ_2^* of H_1, H_2, which are induced by a generator of Z, pulled back to G_1, G_2, and then acting on H_1, H_2 by conjugation.

Since ρ^* is an isomorphism we may assume

$$\rho^*\phi_1^* = \phi_2^*(\rho^* \mid H_1). \tag{3}$$

According to Stallings theorem [1] there is a fibering of M_2 which induces (2). Denote by T_2 the fiber of this map. Cut M_1, M_2, along a fiber, obtaining $T_1 \times I$, $T_2 \times I$. Denote by ϕ_i: $T_i \times 0 \rightarrow T_i \times 1$ the maps which repair these cuts. Clearly ϕ_i induces ϕ_i^* modulo an inner automorphism of H_i.

Now if a homeomorphism ρ: $T_1 \times I \rightarrow T_2 \times I$ can be found satisfying

$$\phi_2(\rho \mid T_1 \times 0) = \rho\phi_1, \tag{4}$$

then ρ defines a homeomorphism from M_1 to M_2.

An algebraic map ρ^* from $\pi_1(T_1)$ to $\pi_1(T_2)$ is already defined, so according to a theorem of Nielson [2], and Mangler [4], there exists a homeomorphism ρ_1: $T_1 \rightarrow T_2$ such that $\rho_1^* = \rho^*$. Now[1] $(\rho_1\phi_1)^* = (\phi_2(\rho_1 \mid T_1 \times 0))^*$. According to a theorem of Baer [3] (for orientable surfaces) and Mangler [4] (for orientable and nonorientable surfaces), the maps $\rho_1\phi_1$ and $\phi_2(\rho_1 \mid T_1 \times 0)$ differ by an isotopy of T_2. Let us call this isotopy h_t. Then $h_0 \circ \rho_1 \circ \phi_1 = \rho_1 \circ \phi_1$, $h_1 \circ \rho_1 \circ \phi_1 = \phi_2(\rho_1 \mid T_1 \times 0)$. Define ρ: $T_1 \times I \rightarrow T_2 \times I$ as follows:

$$\rho(x, t) = (h_t\rho_1, t)$$

then

$$\rho(x, 1) = (h_1\rho_1, 1)$$

$$\rho(x, 0) = (h_0\rho_1, 0) = (\rho_1, 0).$$

So that

$$\rho\phi_1 = h_1\rho_1\phi_1 = \phi_2(\rho_1 \mid T_1 \times 0)$$

but

$$\begin{aligned} \phi_2(\rho \mid T_1 \times 0) &= \phi_2(h_0\rho_1 \mid T_1 \times 0) \\ &= \phi_2(\rho_1 \mid T_1 \times 0) \end{aligned}$$

and so (4) is satisfied and the theorem is proved.

The compact case. As far as the compact nonclosed case is concerned a somewhat different approach may be adopted.

Suppose M_1 is a compact, *orientable*, irreducible, 3-manifold, and $\pi_1(M_1) = G_1$ contains a normal subgroup H_1 such that:

(a) H_1 is finitely generated.

(b) $G_1/H_1 \approx Z$.

(c) $H_1 \not\approx Z_2$.[2]

Having already investigated the case $\partial M_1 = \phi$, we may assume $\partial M_1 \neq \phi$. According to Stallings theorem [1] M_1 is fibered over S^1 with fiber a 2-manifold S_1. Since M_1 is orientable this fibering implies

[1] Modulo an inner automorphism.

[2] This constitutes part of the hypothesis of Theorem 2.

each boundary component of M_1 is a torus. Denote by $T_1, \cdots, T_n$ these boundary tori. Since each boundary torus has a fibering induced in it, we may select curves m_i, l_i in each T_i such that m_i covers S^1 once under the projection of the fibering, and each l_i lies in a fiber. It follows then that m_i, l_i generate π_1 of T_i, but further, it also follows that each m_i, l_i is not homotopic to 0 in M_1. Join each T_i to a base point, b, in M_1 by an arc α_i. Then by the above remarks each natural map $\pi_1(T_i \cup \alpha_i, b) \rightarrow \pi_1(M_1, b)$ is a monomorphism. Consider now the group $\pi_1(M_1, b) = G$ and subgroups $\pi_1(T_i \cup \alpha_i, b) = A_i$. If a different set of arcs α_i be selected, then a set of subgroups $\overline{A}_i$ results, where each $\overline{A}_i$ is a conjugate of A_i. In view of this, we may investigate the topological invariant $(G, [A_1], [A_2], \cdots, [A_n])$, where G is $\pi_1(M_1, b)$ and $[A_i]$ is the conjugacy class containing A_i. Call this invariant the peripheral system of M_1. (See [5] for the source of this invariant.)

THEOREM 2. *Suppose a compact 3-manifold M_2 has peripheral system $(G', [A_1'], [A_2'], \cdots, [A_n'])$, then if there exists an isomorphism $\phi: G \rightarrow G'$, mapping $[A_i]$ onto $[A_i']$, M_1 is homeomorphic to M_2.*

PROOF. By Stallings theorem [1], M_2 is fibered over S^1 with fiber a 2-manifold S_2, where $\pi_1(S_2) = \phi(\pi_1(S_1))$. Now define homeomorphisms Ψ_i: $T_i \cup \alpha_i \rightarrow U_i \cup \beta_i$ (where U_i are the boundary tori of M_2 and β_i are arcs joining U_i to a base point in M_2) such that $\Psi^* = \phi$ for each element x in $\pi_1(T_i \cup \alpha_i)$. This may be done by virtue of [2] and the hypothesis. It is no loss of generality to assume the α_i all lie on one fiber, and similarly the β_i. Nielson's [2] may be slightly generalized (as in [6]) so that a homeomorphism Ψ_{n+1} may be constructed from the fiber containing the α_i to the fiber containing the β_i, satisfying $\Psi_{n+1}^* = \phi$ for elements x in H_1, and agreeing with the Ψ_i on $(T_i \cup \alpha_i) \cap$(fiber containing α_i). Call the homeomorphism now defined on $\partial M_1 \cup$(a fiber), Ψ. Ψ may be extended to a small closed product neighborhood of $\partial M_1 \cup$(a fiber). Denote by N this neighborhood, and, by Ψ the homeomorphism thereon defined. Now $M_1 - (\text{int } N)$ is a solid torus of some genus (being fibered over an interval), and it is easily seen that Ψ^* maps the kernel of the inclusion $\pi_1(\partial(M_1 - \text{int } N)) \rightarrow \pi_1(M_1 - \text{int } N)$ onto the kernel of $\pi_1(\partial(M_2 - \text{int } \Psi(N))) \rightarrow \pi_1(M_2 - \text{int } \Psi(N))$. (The argument is exactly that in [6], with H_1 taking the place of the commutator subgroup.) Hence (as in [6]) Ψ may be extended to all of M_1 and the theorem is proved.

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THE PRODUCT OF A NORMAL SPACE AND A METRIC SPACE NEED NOT BE NORMAL

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An old—and still unsolved—problem in general topology is whether the cartesian product of a normal space and a closed interval must be normal. In fact, until now it was not known whether, more generally, the product of a normal space X and a metric space Y is always normal. The purpose of this note is to answer the latter question negatively, even if Y is separable metric and X is Lindelöf and hereditarily paracompact.

Perhaps the simplest counter-example is obtained as follows: Take Y to be the irrationals, and let X be the unit interval, retopologized to make the irrationals discrete. In other words, the open subsets of X are of the form $U \cup S$, where U is an ordinary open set in the interval, and S is a subset of the irrationals.[2] It is known, and easily verified, that any space X obtained from a metric space in this fashion is normal (in fact, hereditarily paracompact). Now let Q denote the rational points of X, and U the irrational ones. Then in $X \times Y$ the two disjoint closed sets $A = Q \times Y$ and $B = \{(x, x) \mid x \in U\}$ cannot be separated by open sets. To see this, suppose that V is a neighborhood of B in $X \times Y$. For each n, let

$$U_n = \{x \in U \mid (\{x\} \times S_{1/n}(x)) \subset V\},$$

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[1] Supported by an N.S.F. contract.

[2] The usefulness of this space X for constructing counterexamples was first called to my attention, in a different context, by H. H. Corson.