POWER SERIES WITH INTEGRAL COEFFICIENTS

BY DAVID G. CANTOR

Communicated by P. T. Bateman, January 17, 1963

Let $f(z)$ be a function, meromorphic in $|z|<1$, whose power series around the origin has integral coefficients. In [5], Salem shows that if there exists a nonzero polynomial $p(z)$ such that $p(z)f(z)$ is in H^2, or else if there exists a complex number α, such that $1/(f(z)-\alpha)$ is bounded, when $|z|$ is close to 1, then $f(z)$ is rational. In [2], Chamfy extends Salem's results by showing that if there exists a complex number α and a nonzero polynomial $p(z)$, such that $p(z)/(f(z)-\alpha)$ is in H^2, then $f(z)$ is rational. In this paper we show that if $f(z)$ is of bounded characteristic in $|z|<1$ (i.e. the ratio of two functions, each regular and bounded in $|z|<1$), then $f(z)$ is rational. If $f(z)$ is regular in $|z|<1$, then, by [4], $f(z)$ is of bounded characteristic in $|z|<1$, if and only if

$$\limsup_{r\to 1-} \int_0^{2\pi} \log^+ |f(re^{i\theta})|\, d\theta < \infty.$$

Thus any function in any H^p space $(p>0)$ is of bounded characteristic. Hence, since the functions of bounded characteristic form a field, our result includes those of Salem and Chamfy.

Our first lemma gives a necessary condition for a function to be of bounded characteristic in $|z|<1$, in terms of the properties of its Taylor series coefficients.

If $g(z)=\sum_{i=0}^{\infty} a_i z^i$, we denote by $A_r=A_r(g)$ the matrix $\|a_{i+j}\|$, $0\leqq i, j\leqq r$.

LEMMA 1. *Suppose $g(z)$ is of bounded characteristic in $|z|<1$. Then $\det(A_r)\to 0$ as $r\to\infty$. More precisely,* $\lim_{r\to\infty} |\det(A_r)|^{1/r}=0$.

PROOF. By assumption, we may write $g(z)=s(z)/t(z)$, where $s(z)$ and $t(z)$ are bounded analytic functions in $|z|<1$. Suppose that $s(z)=\sum_{i=0}^{\infty} s_i z^i$ and $t(z)=\sum_{i=0}^{\infty} t_i z^i$, and, without loss of generality, that $t_0=1$. We now perform a series of column and row operations on the matrix A_r. Denote its columns from left to right by c_0, c_1, $c_2, \cdots, c_r$. Now, successively, for $j=0, 1, 2, \cdots, r$, replace the column c_{r-j} by $\sum_{i=0}^{r-j} t_i c_{r-j-i}$; then perform the same sequence of operations on the rows. This yields a matrix $D_r=\|d_{mn}\|$, $0\leqq m, n\leqq r$. Since $t_0=1$, $\det(D_r)=\det(A_r)$.

It is easy to verify that

$$d_{mn} = \sum_{i=0}^{m} \sum_{j=0}^{n} t_i t_j a_{m+n-i-j}.$$

Hence d_{mn} is the coefficient of z^{m+n} in

$$\sum_{i=0}^{m} t_i z^i \sum_{j=0}^{n} t_j z^j \sum_{k=0}^{\infty} a_k z^k$$

$$(1) \qquad = \left(t(z) - \sum_{i=m+1}^{\infty} t_i z^i\right)\left(t(z) - \sum_{j=n+1}^{\infty} t_j z^j\right) g(z)$$

$$= \left(t(z) - \sum_{i=m+1}^{\infty} t_i z^i - \sum_{j=n+1}^{\infty} t_j z^j\right) s(z) + g(z) \sum_{i=m+1}^{\infty} t_i z^i \sum_{j=n+1}^{\infty} t_j z^j,$$

since $t(z)g(z) = s(z)$. As the coefficient of z^{m+n} in the last term of (1) is 0, d_{mn} is the coefficient of z^{m+n} in

$$\left(\sum_{j=0}^{m} t_j z^j + \sum_{j=0}^{n} t_j z^j - \sum_{j=0}^{\infty} t_j z^j\right) \sum_{j=0}^{\infty} s_j z^j.$$

Hence

$$d_{mn} = \alpha_{mn} + \alpha_{nm} - \beta_{mn},$$

where $\alpha_{mn} = \sum_{i=0}^{m} t_i s_{m+n-i}$ and $\beta_{mn} = \sum_{i=0}^{m+n} t_i s_{m+n-i}$. Then, by Schwarz's inequality,

$$(2) \qquad |d_{mn}|^2 \leqq 3(|\alpha_{mn}|^2 + |\alpha_{nm}|^2 + |\beta_{mn}|^2).$$

We now show that

$$\sum_{m=0}^{r} \sum_{n=0}^{r} |d_{mn}|^2 = o(r).$$

To do this, it suffices to show that

$$\sum_{m=0}^{r} \sum_{n=0}^{r} |\alpha_{mn}|^2 = o(r),$$

and that

$$\sum_{m=0}^{r} \sum_{n=0}^{r} |\beta_{mn}|^2 = o(r).$$

Now, α_{mn} is the coefficient of z^{m+n} in $\sum_{i=0}^{\infty} t_i z^i \sum_{j=n}^{\infty} s_j z^j$. Hence, by Parseval's equality,

$$(3) \qquad \sum_{m=0}^{\infty} |\alpha_{mn}|^2 = \lim_{\rho \to 1-} \frac{1}{2\pi} \int_0^{2\pi} \left| t(z) \sum_{i=n}^{\infty} s_i z^i \right|^2 d\theta,$$

where $z=\rho e^{i\theta}$. Now $t(z)$ is bounded in $|z|<1$ by, say, T. Thus, again using Parseval's equality, we have, when $|z|=\rho<1$,

$$\frac{1}{2\pi}\int_0^{2\pi}\left|t(z)\sum_{i=n}^{\infty}s_iz^i\right|^2 d\theta \leqq \frac{T^2}{2\pi}\int_0^{2\pi}\left|\sum_{i=n}^{\infty}s_iz^i\right|^2 d\theta = T^2\sum_{i=n}^{\infty}|s_i|^2\rho^{2i}. \tag{4}$$

Put $S_n=\sum_{i=n}^{\infty}|s_i|^2$. Then, as $s(z)$ is bounded, S_0 is finite and $S_n\to 0$ as $n\to\infty$. By (3) and (4), we have

$$\sum_{m=0}^{\infty}|\alpha_{mn}|^2 \leqq T^2S_n.$$

Hence

$$\sum_{n=0}^{r}\sum_{m=0}^{r}|\alpha_{mn}|^2 \leqq T^2\sum_{n=0}^{r}S_n = o(r). \tag{5}$$

Now, β_{mn} is the coefficient of z^{m+n} in the bounded function $s(z)t(z) = \sum_{i=0}^{\infty}u_iz^i$. Then,

$$\sum_{m=0}^{r}|\beta_{mn}|^2 \leqq \sum_{m=0}^{\infty}|\beta_{mn}|^2 = \sum_{i=n}^{\infty}|u_i|^2.$$

Thus,

$$\sum_{n=0}^{r}\sum_{m=0}^{r}|\beta_{mn}|^2 = o(r). \tag{6}$$

Hence, by (2), (5), and (6),

$$\sum_{m=0}^{r}\sum_{n=0}^{r}|d_{mn}|^2 = o(r).$$

We now estimate $\det(D_r)$. By Hadamard's inequality,

$$|\det(D_r)|^2 \leqq \prod_{m=0}^{r}\sum_{n=0}^{r}|d_{mn}|^2. \tag{7}$$

The right hand side of (7) is the $(r+1)$st power of the geometric mean of the quantities $\sum_{n=0}^{r}|d_{mn}|^2$, $0\leqq m\leqq r$. Hence, by the inequality between arithmetic and geometric means

$$|\det(D_r)|^{2/(r+1)} \leqq \frac{1}{r+1}\sum_{m=0}^{r}\sum_{n=0}^{r}|d_{mn}|^2 = o(1).$$

Hence, since $\det(D_r)=\det(A_r)$, we have

$$\lim_{r\to\infty} |\det(A_r)|^{1/r} = 0. \qquad \text{q.e.d.}$$

By a change of variable we obtain

LEMMA 2. *Suppose $g(z)$ is regular at $z=0$, and of bounded characteristic in $|z|<s$. Then $\lim_{r\to\infty} s^r|\det(A_r(g))|^{2/r}=0$.*

THEOREM 1. *Let $f(z)$ be a function of bounded characteristic in $|z|<1$, whose Laurent series around the origin has integral coefficients. Then $f(z)$ is rational.*

PROOF. By multiplying $f(z)$ by a power of z, if necessary, we may assume that $f(z)$ is regular at $z=0$, and has a power series expansion $f(z) = \sum_{i=0}^{\infty} a_i z^i$, where the a_i are integers. By Lemma 1, $\lim_{n\to\infty} \det(A_n(f))=0$. As the a_i are integers, so are the $\det(A_n(f))$. It follows that $\det(A_n(f))=0$ for all large n. But this implies that $f(z)$ is rational, by a theorem by Kronecker [1, p. 138].

COROLLARY. *Let $f(z)$ be a function meromorphic in $|z|<1$, whose Laurent series around the origin has integral coefficients. If there exists a set S of positive capacity, such that for each $\alpha\in S$, the equation $f(z)=\alpha$ has only finitely many solutions in $|z|<1$, then $f(z)$ is rational.*

PROOF. If $f(z)$ satisfies only the second condition, then by a theorem of Frostman [3] or [4, p. 260], $f(z)$ is of bounded characteristic. q.e.d.

Let K be an algebraic number field of degree n over the rationals; denote by $K^{(i)}$, $1\leqq i\leqq n$, the different embeddings of K into the field of complex numbers. If $a\in K$, denote by $a^{(i)}$ the image of a in $K^{(i)}$.

THEOREM 2. *Let $f(z)=\sum_{j=0}^{\infty} a_j z^j$ be a formal power series whose coefficients a_j are algebraic integers in K. Suppose that $f^{(i)}(z) = \sum_{j=0}^{\infty} a_j^{(i)} z^j$ is of bounded characteristic in the disc $|z|<s_i$, $1\leqq i\leqq n$, where $\prod_{i=1}^{n} s_i \geqq 1$. Then $f(z)$ is a rational function.*

PROOF. Put $A_r = A_r(f)$ and $A_r^{(i)} = A_r(f^{(i)})$. By Lemma 2, $s_i^r|\det A_r^{(i)}|^{2/r}\to 0$ as $r\to\infty$. Hence

$$\operatorname{Nm}\det(A_r) = \prod_{i=1}^{n} \det(A_r^{(i)}) \to 0$$

as $r\to\infty$. Since $\operatorname{Nm}\det(A_r)$ is an integer, it is eventually 0. Hence by the theorem of Kronecker (whose proof is valid over any field) [1, p. 138], $f(z)$ is rational.

REFERENCES

1. J. Cassels, *An introduction to Diophantine approximation*, Cambridge Tract 45, Cambridge Univ. Press, Cambridge, 1957.
2. C. Chamfy, *Fonctions méromorphes dans le cercle-unité et leurs series de Taylor*, Ann. Inst. Fourier Grenoble 8 (1958), 237–245.
3. O. Frostman, *Über die defecten Werte einer meromorphen Funktion*, 8 Congr. Math. Scand., Stockholm, 1934.
4. R. Nevanlinna, *Eindeutige Analytische Funktionen*, Springer Verlag, Berlin, 1936.
5. R. Salem, *Power series with integral coefficients*, Duke Math. J. 12 (1945), 153–172.

UNIVERSITY OF WASHINGTON

COMMUTING VECTOR FIELDS ON 2-MANIFOLDS

BY ELON L. LIMA[1]

Communicated by Deane Montgomery, January 28, 1963

We shall consider C^1 vector fields X, Y on a compact 2-manifold M. When the Lie bracket $[X, Y]$ vanishes identically on M, we say that X and Y *commute*. It was shown in [1] that every pair of commuting vector fields on the 2-sphere S^2 has a common singularity. Here we extend this result to all compact 2-manifolds with nonvanishing Euler characteristic.

Our manifolds are connected and may have boundary. The boundary of a compact 2-manifold is either empty or consists of finitely many disjoint circles. Given a C^1 vector field X on a compact manifold M, we tacitly assume that X is tangent to the boundary of M (if it exists). Then the trajectories of X are defined for all values of the parameter, and translation along them provides a (differentiable) action ξ of the additive group R on M. Given $x \in M$, one has $X(x) = 0$ if, and only if, x is a fixed point of ξ, that is, $\xi(s, x) = x$ for all $s \in R$. Let Y be another C^1 vector field on M, generating the action η of R on M. The condition $[X, Y] \equiv 0$ means that ξ and η commute, that is, $\xi(s, \eta(t, x)) = \eta(t, \xi(s, x))$ for all $x \in M$ and $s, t \in R$. Thus the pair X, Y generates an action $\phi: R^2 \times M \to M$ of the additive group R^2 on M, defined by $\phi(r, x) = \xi(s, \eta(t, x)) = \eta(t, \xi(s, x))$ for $x \in M$ and $r = (s, t) \in R^2$. Notice that $x \in M$ is a fixed point of ϕ if, and only if, x is a common singularity of X and Y, that is, $X(x) = Y(x) = 0$. These

[1] Fellow of the John Simon Guggenheim Memorial Foundation. Work partially supported by the National Science Foundation, under NSF-G21514.