OBSTRUCTION THEORIES FOR SMOOTHING MANIFOLDS AND MAPS

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1. **Introduction.** In his papers [5] and [6], James Munkres defines two obstruction theories. The first attacks the problem of smoothing a map, the second that of smoothing a manifold. We present two similar obstruction theories which avoid certain difficulties present in the earlier ones.

The obstruction cochains are defined and their properties stated in §§3 and 4. §2 presents the fundamental result on which the paper is based. In §5 we outline a proof of the conjecture of John Milnor that $\Gamma_{i-1} = \pi_i(B_{PL}, B_O)$. Details will be presented elsewhere.

Some of these results have been obtained independently by Barry Mazur.

2. **The Product Theorem.** A differential manifold will be denoted by an ordered pair M_α, where M is a combinatorial manifold and α is a compatible differential structure on M. (Strictly speaking, M_α is a differential manifold with a distinguished class of smooth triangulations.) If $U \subset M$ is an open set, then $\alpha | U$ and U_α have the obvious meanings. We call α a *smoothing* of M.

Let I be the closed unit interval. Two smoothings α, β of an unbounded combinatorial manifold M are *concordant* if there is a smoothing γ of $M \times I$ such that the boundary of $(M \times I)_\gamma$ is $(M \times 0)_\alpha \cup (M \times 1)_\beta$. (This definition is due to Milnor.)

Suppose that there is a smoothing δ of a neighborhood U of a subcomplex $K \subset M$ such that $(U \times I)_\gamma = U_\delta \times I$. Then we say that α and β are *concordant rel* K; the notation is $M_\alpha \doteq M_\beta$ rel K.

Concerning the relationship between concordance and diffeomorphism, the following results are known:

THEOREM 1.1. (a) *Concordance implies diffeomorphism* (Thom [8], Munkres [7]).

(b) *For spheres, diffeomorphism implies concordance* (Milnor).

(c) *There are smooth manifolds which are diffeomorphic but not concordant.*

For example, let α, β be smoothings of the combinatorial n-sphere S such that S_α and S_β are not concordant. It is known [4] that for large enough m, $S_\alpha \times R^m$ and $S_\beta \times R^m$ are diffeomorphic. However, it follows from the Product Theorem that if $S_\alpha \times R^m \simeq S_\beta \times R^m$, then $S_\alpha \simeq S_\beta$.

THEOREM 2.1 (PRODUCT THEOREM). *Let α be a smoothing of a neighborhood U of a subcomplex K of a combinatorial unbounded manifold M; K may be empty. Let β be a smoothing of $M\times R^n$ such that $(U\times R^n)_\beta = U_\alpha \times R^n$. There is a smoothing γ of M with the following properties*:

(a) $M_\gamma \times R^n \simeq (M\times R^n)_\beta$ rel $K\times R^n$.

(b) *There is a neighborhood V of K in U such that $\gamma | V = \alpha | V$. Moreover, any two smoothings of M satisfying* (a) *and* (b) *are concordant* rel K.

The proof of existence is by induction on n. The case $n=1$ is essentially contained in Theorem 2.5 of [2]. The induction is completed by observing that $M\times R^{n+1} = (M\times R^n)\times R^1$. Uniqueness follows easily from existence.

3. Obstructions to smoothing manifolds. Recall that Γ_k may be defined as the group of oriented diffeomorphism classes of smoothings of the combinatorial k-sphere S^k. The group operation is the formation of the connected sum.

Let M be a combinatorial n-manifold, with a fixed triangulation. Let K be a subcomplex. Denote the i-skeleton of M by M_i. Let α be a smoothing of a neighborhood U of $K\subset M_i$, and let σ^{i+1} be an $(i+1)$-simplex of M with boundary $\partial\sigma^{i+1}$. There is an open regular neighborhood $N(\partial\sigma^{i+1})$ inside U. Now $N(\partial\sigma^{i+1})$ and $S^i\times R^{n-i}$ are combinatorially equivalent, denoted by $N(\partial\sigma^{i+1}) \equiv S^i\times R^{n-i}$. From the Product Theorem (2.1) it follows that there is a unique element $\beta\in\Gamma_i$ such that $N(\partial\sigma^{i+1})_\alpha \simeq (\partial\sigma^{i+1})_\beta \times R^{n-i}$. Put $\beta = C_\alpha(\sigma^{i+1})$.

THEOREM 3.1. *The cochain $C_\alpha \in C^{i+1}(M;\Gamma_i)$ has the following properties*:

(a) *C_α is a cocycle and vanishes on simplices in K.*

(b) *$C_\alpha(\sigma^{i+1}) = 0$ if and only if the smoothing α can be extended over a neighborhood of σ^{i+1}.*

(c) *If α' is a smoothing of a neighborhood of $K\cup M_i$ which agrees with α in a neighborhood of $K\cup M_{i-1}$, then $C_{\alpha'} - C_\alpha$ is a coboundary* mod K, *and every $(i+1)$-coboundary* mod K *is obtained as α' varies.*

(d) *If α' is a smoothing of a neighborhood of $K\cup M_i$ such that $\alpha' \simeq \alpha$, then $C_{\alpha'} = C_\alpha$.*

4. Obstructions to smoothing maps. Let M be a combinatorial manifold and N a differential manifold. A homeomorphism $f\colon M\to N$ is *piecewise regular* if each closed simplex of some rectilinear triangulation of M is mapped diffeomorphically. Two such maps $f_i\colon M\to N$ $(i=0, 1)$ are *concordant rel* K (where $K\subset M$ is a subcomplex) if there is a piecewise regular homeomorphism $G\colon M\times I\to N\times I$ such that for $i=0, 1$ we have $G(x, i)=(f_i(x), i)$, and such that for some neighborhood W of $K\times I$ in $M\times I$, we have $G(x, t)=(f_0(x), t)$ for all $(x, t)\in W$.

The following result is a strengthening of 1.1a.

THEOREM 4.1. *Let α, β be smoothings of M that are concordant* rel K. *Then there is a diffeomorphism $g\colon M_\alpha\to M_\beta$ that is concordant* rel K *to the identity map $M_\alpha\to M_\beta$.*

This theorem translates the problem of concordance of maps into a problem of concordance of differential structures, as follows. Let $f\colon M_\alpha\to V_\beta$ be a piecewise regular homeomorphism. Let $K\subset M$ be a subcomplex, and suppose that f maps a neighborhood of K diffeomorphically; we say f is *smooth near* K. Suppose in particular that f is smooth near $K\cup M_{i-1}$. We ask for a piecewise regular homeomorphism $g\colon M_\alpha\to V_\beta$ which is smooth near $K\cup M_i$, and which is concordant to f rel $K\cup M_{i-1}$. It follows from Theorem 4.1 that this is the case if and only if there is a smoothing γ of $M\times R$ such that $(M\times R)_\gamma$ agrees with $M_\alpha\times R$ in a neighborhood of $M\times(-\infty, 0]\cup(K\cup M_{i-1})\times R$ and with $M_{f^*\beta}$ in a neighborhood of $M\times[1, \infty)$. (Here $f^*\beta$ is the unique smoothing of M such that $f\colon M_{f^*\beta}\to V_\beta$ is a diffeomorphism.)

To define an obstruction cochain, let $\sigma^i\subset M$ be an i-simplex. The smoothing γ described above already exists in a neighborhood of $\sigma^i\times 0\cup\sigma^i\times 1\cup\partial\sigma^i\times I=\partial(\sigma^i\times I)$. The obstruction to extending γ over a neighborhood of $\sigma^i\times I$ lies in $H^{i+1}(\sigma^i\times I, \partial(\sigma^i\times I); \Gamma_i)\approx\Gamma_i$, as described in §3. Let this obstruction be denoted by $C_f(\sigma^i)$. Thus $C_f\in C^i(M; \Gamma_i)$.

THEOREM 4.2. *The cochain C_f has the following properties*:

(a) *C_f is a cocycle and vanishes on simplices in K.*

(b) *$C_f(\sigma^i)=0$ if and only if f is concordant* rel $K\cup M_{i-1}$ *to a piecewise regular homeomorphism $g\colon M_\alpha\to V_\beta$ which is smooth near $K\cup M_{i-1}\cup\sigma^i$.*

(c) *If $f'\colon M_\alpha\to V_\beta$ is a piecewise regular homeomorphism which agrees with f in a neighborhood of $K\cup M_{i-2}$, then $C_{f'}-C_f$ is a coboundary* mod K, *and every coboundary* mod K *is obtained as f' varies.*

(d) *If $f'\colon M_\alpha\to V_\beta$ is a piecewise regular homeomorphism which is smooth near $K\cup M_{i-1}$, and f' is concordant to f by a piecewise regular* *homeomorphism* $G: M_\alpha \times I \to V_\beta \times I$ *which is smooth near* $(K \cup M_{i-1}) \times I$, *then* $C_{f'} = C_f$.

It is possible to define C_f more directly, and to prove that when $C_f(\sigma^i) = 0$, the map $g: M_\alpha \to V_\beta$ appearing in part (b) of the theorem can be chosen so as to approximate f, and to agree with f outside a given neighborhood of $M \cup K_i$ as well as in a neighborhood of $M \cup K_{i-1}$.

This obstruction theory can easily be modified so as to apply to piecewise regular *local* homeomorphisms.

5. **On the groups Γ_k.** Milnor [3] has defined a space B_{PL}, which is a classifying space for stable equivalence classes of piecewise linear microbundles.

Every orthogonal bundle (i.e., n-plane bundle with structural group $O(n)$) determines an underlying microbundle, hence we may consider B_O as a subcomplex of B_{PL}, where B_O is classifying space for stable equivalence classes of orthogonal bundles. Milnor conjectured that $\Gamma_{i-1} \approx \pi_i(B_{PL}, B_O)$.

THEOREM 5.1. *Let ψ assign to each microbundle ξ over S^i the obstruction $\psi(\xi) \in \Gamma_{i-1}$ to smoothing a regular neighborhood of the zero section of the total space of ξ. Then ψ induces an isomorphism between $\pi_i(B_{PL}, B_O)$ and Γ_{i-1}.*

PROOF. We represent an element of $\pi_i(B_{PL})$ by a microbundle $\xi = (p, E, S^i, j)$. Here $p: E \to S^i$ is the projection and $j: S^i \to E$ is the zero cross section so that pj is the identity map of S^i. It is easy to prove, using 3.1, 4.2 and standard techniques of obstruction theory, that if M is a $(k-1)$-connected combinatorial manifold, then there is a unique smoothing of a neighborhood of M_{k-1} up to concordance, where two smoothings are identified if they agree on a common subneighborhood. Therefore the first obstruction to smoothing M is a well-defined cohomology class in $H^k(M; \Gamma_{k-1})$. Moreover, M and $M \times R$ have the same obstruction class. Therefore $\psi(\xi)$ is well defined, and is unchanged if a trivial bundle is added to ξ. It is easy to see that ψ is a homomorphism.

We can identify ξ with the restriction of $p^*(\xi)$ to $j(S^i)$, since p is a homotopy equivalence on a regular neighborhood of $j(S^i)$. If ξ comes from an orthogonal bundle, then E can be smoothed, and so ψ vanishes on the image of $\pi_i(B_O)$ in $\pi_i(B_{PL})$. On the other hand, if $\psi(\xi)$ vanishes then E can be smoothed, and the tangent microbundle τ_E of E comes from an orthogonal bundle. Since $p^*\xi \oplus p^*\tau_{S^i} = \tau_E$, and

τ_{S^i} is stably trivial, we see that ξ comes from $\pi_i(B_O)$ if $\psi(\xi)=0$.

Using the exactness of the homotopy sequence of (B_{PL}, B_O), we see that ψ induces a monomorphism from the image of $\pi_i(B_{PL}) \to \pi_i(B_{PL}, B_O)$ into Γ_{i-1}. It remains to prove that $\psi\colon \pi_i(B_{PL}) \to \Gamma_{i-1}$ is onto, and that $\pi_i(B_{PL}) \to \pi_i(B_{PL}, B_O)$ is onto. This last fact is due to Milnor, who proved it using other methods. To prove it we use exactness and show that if an orthogonal bundle η is stably trivial as a microbundle, then η is stably trivial as an orthogonal bundle. A neighborhood E_0 of the zero section of η is combinatorially equivalent to $S^i \times R^n$. Give E_0 a smoothing α, so that η is a differential bundle. By the Product Theorem, E_0 is diffeomorphic to $S^i_\alpha \times R^n$ where α is a smoothing of S^i. Since Adams has proved that any smooth homotopy sphere is a π-manifold (cf. [1]), E_0 is parallelizable. It follows easily that η is stably trivial.

It remains to prove that ψ maps $\pi_i(B_{PL})$ onto Γ_{i-1}. The following argument was suggested by Milnor, and replaces a more complicated one of the author. If $\alpha \in \Gamma_{i-1}$ is a smoothing of S^{i-1}, then $S^{i-1}_\alpha \times I$ has a trivial tangent bundle. Also $S^{i-1} \times I$ with the ordinary differential structures has a trivial tangent bundle, and the underlying microbundles are identical. This gives us a microbundle equivalence $f\colon S^{i-1} \times R^i \to S^{i-1} \times R^i$ which produces a microbundle ξ over S^i when two copies of $D^i \times R^i$ are glued together by f. It is easy to see that $\psi(\xi)=\alpha$, completing the proof of the theorem.

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