DIFFERENTIAL INQEUALITIES

BY RAYMOND M. REDHEFFER[1]

Communicated by Lipman Bers, August 14, 1962

Let B denote a bounded region in Euclidean n-space, with boundary ∂B and closure $\bar{B}$. We write $P = x = (x_1, x_2, \cdots, x_n) \in \bar{B}$, $u_i = \partial u/\partial x_i$, $u_{ij} = \partial^2 u/\partial x_i \partial x_j$, and similarly for v, c and y. The normal derivative u_ν is understood in the sense of Walter, namely:

$$u_\nu(P_0) = \limsup \left[u(P_k) - u(P_0)\right] \left| P_k - P_0 \right|^{-1}$$

where $P_k \in B$, $P_0 \in \partial B$, and $P_k \to P_0$ in such a way that

$$(P_k - P_0) \left| P_k - P_0 \right|^{-1}$$

tends to a fixed vector, ν. We have $u = u(x)$, $v = v(x)$, and

$$Tu = \phi(x) - f(x, u, u_i, u_{ij}), \qquad x \in B,$$

$$Ru = \mathcal{H}(x) - k(x, u, u_\nu), \qquad x \in \partial B.$$

Independent variables are denoted by the letter s. The letter p means "+" or "−," and has the same meaning in hypothesis and conclusion. We suppose ϵ^p and δ^p to be nonnegative constants. The statement "$f(x, v, v_i, v_{ij} \uparrow)$ is monotone" means that

$$p[f(x, v, v_i, v_{ij}) - f(x, v, v_i, s_{ij})] \geqq 0$$

when the matrix $p[(v_{ij}) - (s_{ij})] \geqq 0$, $p = \pm$. Other assertions of monotony are interpreted similarly. We assume $u \in C^{(2)}$, $v \in C^{(2)}$ in B and $u \in C$, $v \in C$ in $\bar{B}$, although discontinuities can be allowed as in [2].

It is convenient to write $v' = (v, v_i, v_{ij})$, a vector of $1 + n + n^2$ components, and similarly for u, s, and y. Also $f' = (f_u, f_{ui}, f_{ui_j})$ with argument (x, v') or (x, s'), as the case may be. Similarly, $k' = (k_u, k_{u_\nu})$. The statement "f' is continuous in the neighborhood of v" means that there is an $h > 0$ such that $f'(x, s')$ is continuous for $|s' - v'| < h$. Other statements of this kind are understood similarly.

THEOREM I. *Let $k(x, u\downarrow, u_\nu)$ be strictly monotone, let $k(x, v, v_\nu \uparrow)$ be monotone, and let f' be continuous in the neighborhood of v. Suppose further:*

(i) *$f(x, u\downarrow, u_i, u_{ij})$ is monotone, and $f(x, s, s_i, s_{ij}\uparrow)$ is monotone in the neighborhood of v.*

[1] Fulbright Research Scholar at the University of Hamburg.

(ii) *To every compact subset* $S \subset B$ *corresponds a function* $c(x) \in C^{(2)}$ *such that*

$$\sum f_{v_i}(x, v')c_i + \sum f_{v_{ij}}(x, v')c_{ij} > 0$$

at those points of S (*if there are any*) *at which*

$$f_v(x, v') = \sum f_{v_{ij}}(x, v')c_i c_j = 0.$$

Then $p(Tu - Tv) \leqq 0$ *and* $p(Ru - Rv) \leqq 0 \Rightarrow p(u - v) \leqq 0$.

To prove the theorem let $y \in C$, $y_0 \in C$, and suppose the conclusion violated. For small $h > 0$ the function $w = p(u - v) - hy$ has an interior maximum and at that point $w_0 = p(u - v) - hy_0 > 0$. We choose $y = -\mu\{c(x)\}$ for a suitable function μ, and $y_0 = -f_v(x, v')$.

Let $2c(x) = r^2 r_0^{-1} - r_0$ where r is the distance to a fixed point P_0 and where r_0 is constant. The unit normal, ν, to the sphere $r = |r_0|$ is $\nu_i = c_i$. A point P is called a sphere-point (r_0, ν) of the set $u = v$ if P is on the sphere $r = |r_0|$, if $u = v$ at P, and if there is a neighborhood N of P such that $p(u - v) < 0$ in those points of N at which $c(x) > 0$. Thus, when $r_0 > 0$ the set $u = v$ lies locally inside a sphere of radius r_0 and outer normal ν, whereas if $r_0 < 0$ the set lies locally outside a sphere of radius $|r_0|$ and inner normal ν. The following result affords a smooth transition from the weak to the strong maximum principle:

THEOREM II. *Let* f' *be continuous in the neighborhood of* v, *let* $f(x, s, s_i, s_{ij} \uparrow)$ *be monotone in the neighborhood of* v, *and suppose further*:

(i) *At the point* $P \in B$, *either*

$$\sum \nu_i f_{v_i}(x, v') + r_0^{-1} \sum f_{v_{ii}}(x, v') > 0$$

or

$$\sum f_{v_{ij}}(x, v')\nu_i \nu_j > 0.$$

(ii) *In a neighborhood of* P, $p(Tu - Tv) \leqq 0$.

Conclusion: P *is not a sphere-point* (r_0, ν) *of the set* $u = v$.

The Fréchet derivative is

$$\lim_{h \to 0} [T(s + hy) - T(s)]h^{-1} = -f'(x, s) \cdot y' \equiv L(s)y$$

where $L(s)$ is, for each s, a linear operator on y. Similarly,

$$\lim_{h \to 0} [R(s + hy) - R(s)]h^{-1} = M(s)y$$

where $M(s)$ is linear. We say that the pair of operators (L, M) belongs to the class (E, D, A) if E, D, and A are positive constants such that the problem

$$Ly \geqq E, \quad x \in B; \quad My \geqq D, \quad x \in \partial B$$

has a solution $y \in C^{(2)}$, $0 \leqq y \leqq 1$, $\|y_i\| + \|y_{ij}\| \leqq A$.

THEOREM III. *Let $f'(x, s')$ and $k'(x, s, s_\nu)$ be uniformly equicontinuous in s and let* $\sup |f_v(x, v')| < \infty$, $\sup |k_v(x, v, v_\nu)| < \infty$. *Suppose further for all s:*

(i) *The matrix* $[f_{s_{ij}}(x, u, u_i, s_{ij})] \geqq 0$, *and* $k_{s_\nu}(x, u, s_\nu) \geqq 0$.

(ii) $[L(s), M(s)] \in (E, D, A)$.

Conclusion: $p(Tu - Tv) \leqq \epsilon^p$ *and* $p(Ru - Rv) \leqq \delta^p \Rightarrow p(u - v) \leqq \max((\epsilon^p/E), (\delta^p/D))$.

The proof follows by constructing a suitable family of solutions $y(x, \xi)$ of

$$p[T(v + py) - Tv] > \epsilon^p, \qquad p[R(v + py) - Rv] > \delta^p,$$

and using the fundamental theorem of Nagumo [3].

Let $c(x) \in C^{(2)}$ be a fixed function with $\inf c(x) = 0$, $\sup \|c_i(x)\| = 1$. The constants $C = \sup c(x)$, $C_2 = \sup \|c_{ij}(x)\|$ measure the size of B with respect to c. The function

$$U^p(\alpha, \beta) = \inf p[f(x, u, u_i, u_{ij} + p\alpha c_{ij} + p\beta c_i c_j) - f(x, u, u_i, u_{ij})]$$

for $\alpha \geqq 0$, $\beta \geqq 0$ measures the influence of the second-derivative terms in f. We write V instead of U when $v(x)$ instead of $u(x)$ occurs on the right. The influence of the first-derivative terms is expressed by

$$p[f(x, u, u_i, s_{ij}) - f(x, v, v_i, s_{ij})] \leqq G^p(S_2, \|u_i - v_i\|) \quad \text{for } p(u - v) > 0$$

where $S_2 = \sup \|s_{ij}\|$, and where G^p is continuous and monotone in both arguments. For simplicity let

$$Ru = u - k(x, u_\nu), \qquad k(x, v_\nu + s) - k(x, v_\nu) \leqq \gamma(|s|)$$

where γ is continuous and increasing. Under these conditions we have:

THEOREM IV. *Let $f(x, u, u_i, s_{ij} \uparrow)$ and $k(x, u_\nu \uparrow)$ be monotone and suppose that $\eta(s)$, for $0 < s < C$, is a positive nondecreasing solution of the differential inequality*

$$U^p(\eta, \eta') > \epsilon^p + G^p(V_2, \eta), \qquad V_2 = \sup\|v_{ij}\|,$$

or of the inequality

$$V^{-p}(\eta, \eta') > \epsilon^p + G^p(V_2 + \eta' + \eta C_2, \eta).$$

Then $p(Tu - Tv) \leqq \epsilon^p$ and $p(Ru - Rv) \leqq \delta^p$ implies

$$p(u-v) \leqq \delta^p + \gamma[\eta(C)] + \int_{c(x)}^{C} \eta(s)ds.$$

The proof follows by setting $\mu'(s)=\eta(s)$, $y=m-\mu[c(x)]$, where m is a constant so chosen that the function $p(u-v)-y$ does not assume a positive maximum on ∂B.

BIBLIOGRAPHY

1. L. Nirenberg, *A strong maximum principle for parabolic equations*, Comm. Pure Appl. Math. **6** (1953), 167–177.
2. R. M. Redheffer, *Bemerkungen über Monotonie und Fehlerabschätzung bei nichtlinearen partiellen Differentialgleichungen*, Arch. Rational Mech. Anal. **10** (1962), 427–457.
3. S. Simoda, and M. Nagumo, *Sur la solution bornée de l'équation aux dérivées partielles du type elliptique*, Proc. Japan Acad. **27** (1951), 334–339.

UNIVERSITY OF HAMBURG

COMPLETE LOCALLY AFFINE SPACES AND ALGEBRAIC HULLS OF MATRIX GROUPS

BY LOUIS AUSLANDER[1]

Communicated by I. M. Singer, November 27, 1962

Let M be a complete Riemann manifold with curvature and torsion zero. If $\pi_1(M)$ denotes the fundamental group of M, then Bieberbach [3; 4] proved that $\pi_1(M)$ contains an abelian normal subgroup of finite index. Moreover, if M is compact then M is covered by a torus.

In recent years the study of general affine connections has led to the study of the following problem: How can one classify the manifolds which possess a complete affine connection with curvature and torsian zero? Such manifolds will be called complete locally affine spaces.

It was Zassenhaus [6] who first gave a general setting to the Bieberbach theorem. He showed a special case of the following theorem:

THEOREM 1. *Let G be a connected Lie group with its radical R simply connected, $\rho: G \to G/R$ the projection, and L a closed subgroup of G. If the identity component L_0 of L is solvable, then the identity component of the closure of $\pi_1(L)$ is solvable.*

This theorem in this generality is due to H. C. Wang [5] and his

[1] With partial support from the N. S. F.