RELATIONS BETWEEN STIEFEL-WHITNEY CLASSES OF MANIFOLDS

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1. **Introduction.** Let M be a C^∞-manifold and let $\tau_M\colon M\to BO$ be the classifying map of its stable tangent bundle. Recall that $H^*(BO; Z_2)$ is a polynomial algebra on the Whitney classes $W_1, W_2, \cdots, W_n, \cdots$. Define the ideal $I_n \subset H^*(BO; Z_2)$ of relations between Stiefel-Whitney classes of manifolds of dimension n as follows:

$$I_n = \bigcap \operatorname{Ker} \tau_M^*$$

where M ranges over all n-dimensional, compact, connected, C^∞-manifolds without boundary.

Let I_n^k denote the elements of I_n of dimension k. E. H. Brown [2] and R. Stong have shown that $I_n^k=0$ if $k\leqq n/2$. A. Dold [3] has calculated I_n^n. In this paper we compute I_n^k for all n and k and furthermore show, in a sense to be made precise in §3, that all of these relations are algebraic in character. In §2 we give the preliminary definitions necessary for the statement of our results and in §3 we give these results.

2. **Right action of the Steenrod algebra.** Let

$$H = \sum_{k=0}^{n} H^k$$

be a graded commutative algebra with unit over Z_2 which is of finite type. Assume A, the mod 2 Steenrod algebra, acts on the left of H as a Hopf algebra (see [4]). This means that the Cartan formula holds, $Sq^i(h)=h^2$ if $\dim(h)=i$ and $Sq^i(h)=0$ if $\dim(h)<i$. Furthermore assume H satisfies Poincaré duality. That is, $H^n\approx Z_2$ and $h\in H^i$ is zero if and only if $hh'=0$ for all $h'\in H^{n-i}$. Such an algebra will be called a Poincaré algebra. Following Adams [1], we define a right operation of A on H by the condition:

$$ha\cdot h' = h\cdot ah'$$

for all $h'\in H^{n-i-j}$ where $h\in H^i$ and $a\in A_j$. Define

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$$v_i = (1)Sq^i,$$

$$W_i = \sum_{j=0}^{i} Sq^i(v_{i-j}),$$

$$\overline{W}_i = \sum_{j=0}^{i-1} \overline{W}_j W_{i-j}, \qquad \overline{W}_0 = 1.$$

It is not difficult to prove the following theorem.

THEOREM 2.1. $(h)\chi(Sq^i) = \sum_{j=0}^{i} \overline{W}_j \cdot Sq^{i-j}(h)$ *where* χ *is the canonical anti-automorphism of* A.

Suppose U is a graded commutative algebra with unit over Z_2 on which A acts on the left as a Hopf algebra. Let $u_i \in U^i$, $i = 0, 1, 2, \cdots$, where $u_0 = 1$. Following Theorem 2.1 we may attempt to define a right action of A on U by the formula:

$$(u)\chi(Sq^i) = \sum_{j=0}^{i} u_j \cdot Sq^{i-j}(u). \tag{2.2}$$

In general this formula will not be consistent with the Adem relations.

THEOREM 2.3. *The formula* (2.2) *makes* U *into a right module over* A *if and only if the* u_i *satisfy the Wu formulae, i.e.*

$$Sq^r(u_i) = \sum_{t=0}^{r} \binom{i-r+t-1}{t} u_{r-t}u_{i+t}.$$

COROLLARY 2.4. *If* $\overline{W}_i = u_i$, (2.2) *makes* $H^*(BO;\, Z_2)$ *into a right module over* A.

COROLLARY 2.5. *If* A *acts on the right of* U *according to the formula* (2.2), *then there is a unique algebra homomorphism* $\tau_U: H^*(BO;\, Z_2) \rightarrow U$ *which is equivariant with respect to the right and left actions of* A.

3. **Relations between Stiefel-Whitney classes.** Let $S \subset H^*(BO; Z_2)$. Define $I_n(S, \text{geom}) = \bigcap \operatorname{Ker} \tau_M^*$ where M runs over all n-dimensional, compact, C^∞-manifolds without boundary such that $\tau_M^*(S) = 0$. Note $I_n(\phi, \text{geom}) = I_n$. Similarly, define

$$I_n(S, \text{alg}) = \bigcap \operatorname{Ker} \tau_H$$

where H ranges over all n-dimensional Poincaré algebras such that $\tau_H(S) = 0$. Clearly $I_n(S, \text{alg}) \subset I_n(S, \text{geom})$.

Let $F_n \subset H^*(BO;\, Z_2)$ be the Z_2 module generated by $H^j(BO;\, Z_2)Sq^i$ for all i and j such that $2i > n-j$. Note that $F_n \subset I_n(\phi, \text{alg})$, for if $x \in H^j(BO;\, Z_2)$ and $2i > n-j$, $\tau_H((x)Sq^i) \cdot h' = \tau_H(x) \cdot Sq^i(h') = 0$ for all $h' \in H^{n-i-j}$. Our main theorem is the following:

THEOREM 3.1. *$I_n = F_n$.*

COROLLARY 3.2. *$I_n = I_n(\phi$, alg).*

COROLLARY 3.3.

(a) *$I_n^k = 0$ if $k \leqq n/2$.*

(b) *$I_n^{[n/2]+1}$ is the Z_2 module generated by $(1)Sq^{[n/2]+1}$.*

(c) *$I_n^{[n/2]+2}$ is the Z_2 module generated by $(1)Sq^{[n/2]+2}$, $W_1((1)Sq^{[n/2]+1})$ and $Sq^1((1)Sq^{[n/2]+1})$.*

(d) *I_n^n is the Z_2 module generated by $(Sq^i + (1)Sq^i \cdot)H^{n-i}(BO; Z_2)$ for $i = 1, 2, \cdots, n$.*

REMARK. 3.3(a) is the theorem of Brown and Stong and 3.3(d) is the theorem of Dold [3].

REMARK. I_n contains the smallest ideal containing $(1)Sq^i$, $i > n/2$, which is closed under the right and left actions of A but this ideal does not equal I_n.

$I_n(S$, alg) may be characterized in the following fashion. Let $J(S)_q \subset H^*(BO; Z_2) \otimes H^*(K(Z_2, q), Z_2)$ be the ideal generated by $S \otimes 1$. Let $L_{n,q} \subset H^*(BO; Z_2) \otimes H^*(K(Z_2, q), Z_2)$ be the Z_2 module generated by all elements of the form:

$$\sum_{=0}^{i} Sq^j(u) \otimes Sq^{i-j}(x) + (1)Sq^i \cdot u \otimes x$$

where $i + \dim u + \dim x = n$. Let $\iota^q \in H^q(K(Z_2, q), Z_2)$ be the canonical generator.

THEOREM 3.4. *$u \in I_n^k(S$, alg) if and only if $u \otimes \iota^{n-k} \in L_{n,n-k} + J(S)_{n-k}$.*

REMARK. It is not true that $I_n(\{W_1\}$, geom) is the ideal generated by I_n and W_1.

REMARK. It is not true that for all S, $I_n(S$, geom$) = I_n(S$, alg).

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